# Fermionic representation for basic hypergeometric functions related to Schur polynomials

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**Abstract**

We present the fermionic representation for the $q$-deformed hypergeometric functions related to Schur polynomials considered by S.Milne [12]. For $q = 1$ these functions are also known as hypergeometric functions of matrix argument which are related to zonal spherical polynomials for $GL(N, C)/U(N)$ symmetric space. We show that these multivariable hypergeometric functions are tau-functions of the KP hierarchy. At the same time they are the ratios of Toda lattice tau-functions considered by Takasaki in [16], [21] evaluated at certain values of higher Toda lattice times. The variables of the hypergeometric functions are related to the higher times of those hierarchies via Miwa change of variables. The discrete Toda lattice variable shifts parameters of hypergeometric functions. Hypergeometric functions of type ${}_pF_s$ can be also viewed as group 2-cocycle for the $\Psi$DO on the circle of the order $p - s \leq 1$ (the group times are higher times of TL hierarchy and the arguments of hypergeometric function). We get the determinant representation and the integral representation of special type of KP tau-functions, these results generalize some of Milne's results in [12]. We write down a system of linear differential and difference equations for these tau-functions (string equations). We present also fermionic representation for special type of Gelfand-Graev hypergeometric functions.

# Contents



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# Introduction

Hypergeometric functions play an important role both, in physics and in mathematics [3]. Many special functions and polynomials (such as $q$-Askey-Wilson polynomials, $q$-Jacobi polynomials, $q$-Gegenbauer polynomials , $q$-Racah polynomials , $q$-Hahn polynomials , expressions for Clebsch-Gordan coefficients) are just certain hypergeometric functions evaluated at special values of parameters. In physics hypergeometric functions and their $q$-deformed counterparts sometimes play the role of wave functions and correlation functions for quantum integrable systems. In the present paper we shall construct hypergeometric functions as tau-functions $\tau$ of the Kadomtsev-Petviashvili (KP) hierarchy of equations. It is interesting that the KP equation

$$4\partial_{t_1}\partial_{t_3}u = \partial_{t_1}^4 u + 3\partial_{t_2}^2 u + 3\partial_{t_1}^2 u^2 \quad (u = 2\partial_{t_1}^2 \log\tau), \tag{0.0.1}$$

which originally served in plasma physics [4] now plays a very important role both, in physics (see [5]; see review in [6] for modern applications) and in mathematics. The peculiarity of Kadomtsev-Petviashvili equation appeared in the paper [7] where L-A pair of KP equation was

presented, and mainly in the paper of V.E.Zakharov and A.B.Shabat in 1974 where this equation was integrated by the dressing method. Actually it was the paper [1] where so-called hierarchy of higher KP equations appeared. Another very important equation is the two-dimensional Toda lattice (TL) integrated first in [8]. In the present paper we use these equations to construct hypergeometric functions which depend on many variables, these variables are KP and Toda lattice higher times. Here we shall use the general approach to integrable hierarchies of Kyoto school [2], see also [10, 9]. Especially a set of papers about Toda lattice [15, 16, 17, 18, 19, 20, 21] is important for us.

About the structure of the paper. To learn the main result, which is the fermionic representation of the hypergeometric functions, one needs read only Sections 1 and Subsections 2.1 and 2.2 - to learn the notations, and then read Subsection 3.1, and Examples 3-6. The linear equations (constraints) for the tau-function, which generalize familiar Gauss equation for the well-known Gauss hypergeometric function see in Subsection 3.7. For determinant representation and integral representation see Subsections 3.8 and 3.9. For hypergeometric function as group two-cocycle see the Remark in the end of Appendix "Gauss factorization problem etc.". All other material is just setting the topic into the theory of integrable systems.

Few words about notations. The symbols $*$ and $^-$ do not denote the complex conjugation. Symbol $'$ does not denote the derivative. Bold $\mathbf{n}$ stands for partitions. Bold $\mathbf{t}$ and $\mathbf{t}^*$ stand for collections of KP and TL higher time variables. Bold $\mathbf{x}_{(N)}$ and $\mathbf{y}_{(N)}$ stand for Miwa variables.

# 1 Milne's hypergeometric series

## 1.1 Ordinary hypergeometric functions

First let us remember that generalized hypergeometric function of one variable $x$ is defined as

$$ {}_pF_s\left(a_1,\ldots,a_p;b_1,\ldots,b_s;x\right)=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_s)_n}\frac{x^n}{n!}. \tag{1.1.1}$$

Here $(a)_n$ is Pochhammer's symbol:

$$ (a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}=a(a+1)\cdots(a+n-1). \tag{1.1.2}$$

Given number $q$, $|q|<1$, the so-called basic hypergeometric series of one variable is defined as

$$ {}_p\Phi_s\left(a_1,\ldots,a_p;b_1,\ldots,b_s;q,x\right)=\sum_{n=0}^{\infty}\frac{(q^{a_1};q)_n\cdots(q^{a_p};q)_n}{(q^{b_1};q)_n\cdots(q^{b_s};q)_n}\frac{x^n}{(q;q)_n}. \tag{1.1.3}$$

Here $(q^a,q)_n$ is $q$-deformed Pochhammer's symbol:

$$ (q^a;q)_0=1,\quad (q^a;q)_n=(1-q^a)(1-q^{a+1})\cdots(1-q^{a+n-1}). \tag{1.1.4}$$

Both series converge for all $x$ in case $p<s+1$. In case $p=s+1$ they converge for $|x|<1$. We refer these well-known hypergeometric functions as ordinary hypergeometric functions.

## 1.2 The multiple basic hypergeometric series related to Schur polynomials

There are several well-known different multivariable generalizations of hypergeometric series of one variable [14, 13]. Let $|q|<1$ and let $\mathbf{x}_{(N)}=(x_1,\ldots,x_N)$ be indeterminates. Let

$s_{\mathbf{n}}(x_1, x_2, ..., x_N)$ be the Schur polynomial corresponding to a partition $\mathbf{n}$ [26]. $s_{\mathbf{n}}(x_1, x_2, ..., x_N)$ is a symmetric function of variables $x_k$. The multiple basic hypergeometric series related to Schur polynomials were introduced by S.Milne [12] as

$$
{}_p\Phi_s\left(a_1, \ldots, a_p; b_1, \ldots, b_s; q, \mathbf{x}_{(N)}\right) = \sum_{\substack{\mathbf{n} \\ l(\mathbf{n})\leq N}} \frac{(q^{a_1};q)_{\mathbf{n}} \cdots (q^{a_p};q)_{\mathbf{n}}}{(q^{b_1};q)_{\mathbf{n}} \cdots (q^{b_s};q)_{\mathbf{n}}} \frac{q^{n(\mathbf{n})}}{H_{\mathbf{n}}(q)} s_{\mathbf{n}}\left(\mathbf{x}_{(N)}\right), \qquad (1.2.1)
$$

where the sum is over all different partitions $\mathbf{n} = (n_1, n_2, \ldots, n_r)$, where $n_1 \geq n_2 \geq \cdots \geq n_r$, $r \leq |\mathbf{n}|$, $|\mathbf{n}| = n_1 + \cdots + n_r$ and whose length $l(\mathbf{n}) = r \leq N$. Schur polynomial $s_{\mathbf{n}}\left(\mathbf{x}_{(N)}\right)$, with $N \geq l(\mathbf{n})$, is a symmetric function of variables $\mathbf{x}_{(N)}$ and defined as follows [26]:

$$
s_{\mathbf{n}}(\mathbf{x}_{(N)}) = \frac{a_{\mathbf{n}+\delta}}{a_\delta}, \quad a_{\mathbf{n}} = \det(x_i^{n_j})_{1\leq i,j\leq N}, \quad \delta = (N-1, N-2, \ldots, 1, 0). \qquad (1.2.2)
$$

Coefficient $(q^c; q)_{\mathbf{n}}$ associated with partition $\mathbf{n}$ is expressed in terms of the $q$-deformed Pochhammer's Symbols $(q^c; q)_n$ (1.1.4):

$$
(q^c; q)_{\mathbf{n}} = (q^c; q)_{n_1}(q^{c-1}; q)_{n_2} \cdots (q^{c-l+1}; q)_{n_l}. \qquad (1.2.3)
$$

The multiple $q^{n(\mathbf{n})}$ defined on the partition $\mathbf{n}$:

$$
q^{n(\mathbf{n})} = q^{\sum_{i=1}^{N}(i-1)n_i}, \qquad (1.2.4)
$$

and $q$-deformed 'hook polynomial' $H_{\mathbf{n}}(q)$ is

$$
H_{\mathbf{n}}(q) = \prod_{(i,j)\in\mathbf{n}} \left(1 - q^{h_{ij}}\right), \quad h_{ij} = (n_i + n'_j - i - j + 1), \qquad (1.2.5)
$$

where $\mathbf{n}'$ is the conjugated partition (for the definition see [26]). For $N = 1$ we get (1.1.3). Another generalization of hypergeometric series is so-called hypergeometric function of matrix argument $\mathbf{X}$ with indices $\mathbf{a}$ and $\mathbf{b}$ [14]:

$$
{}_pF_s\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_s \end{matrix} \middle| \mathbf{X}\right) = \sum_{\mathbf{n}} \frac{(a_1)_{\mathbf{n}} \cdots (a_p)_{\mathbf{n}}}{(b_1)_{\mathbf{n}} \cdots (b_s)_{\mathbf{n}}} \frac{Z_{\mathbf{n}}(\mathbf{X})}{|\mathbf{n}|!}. \qquad (1.2.6)
$$

Here $\mathbf{X}$ is a Hermitian $N \times N$ matrix, and $Z_{\mathbf{n}}(\mathbf{X})$ is zonal spherical polynomial for the symmetric space $GL(N, C)/U(N)$, see [14, 13]. Let us note that in the limit $q \to 1$ series (1.2.1) coincides with (1.2.6), see [13].

## 1.3 Hypergeometric series of double set of arguments

The formula

$$
{}_p\Phi_s\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_s \end{matrix} \middle| q, \mathbf{x}_{(N)}, \mathbf{y}_{(N)}\right) =
$$

$$
\sum_{\substack{\mathbf{n} \\ l(\mathbf{n})\leq N}} \frac{(q^{a_1};q)_{\mathbf{n}} \cdots (q^{a_p};q)_{\mathbf{n}}}{(q^{b_1};q)_{\mathbf{n}} \cdots (q^{b_s};q)_{\mathbf{n}}} \frac{q^{n(\mathbf{n})}}{H_{\mathbf{n}}(q)} \frac{s_{\mathbf{n}}\left(\mathbf{x}_{(N)}\right) s_{\mathbf{n}}\left(\mathbf{y}_{(N)}\right)}{s_{\mathbf{n}}\left(1, q, q^2, ..., q^{N-1}\right)} \qquad (1.3.1)
$$

defines the multiple basic hypergeometric function of two sets of variables which was also introduced by S.Milne, see [12], [13].

Another generalization of hypergeometric series is so-called hypergeometric function of matrix arguments $\mathbf{X},\mathbf{Y}$ with indices $\mathbf{a}$ and $\mathbf{b}$:

$$ {}_p\mathcal{F}_s\left(a_1,\dots,a_p;b_1,\dots,b_s;\mathbf{X},\mathbf{Y}\right)=\sum_{\mathbf{n}}\frac{(a_1)_{\mathbf{n}}\cdots(a_p)_{\mathbf{n}}}{(b_1)_{\mathbf{n}}\cdots(b_s)_{\mathbf{n}}}\frac{Z_{\mathbf{n}}(\mathbf{X})Z_{\mathbf{n}}(\mathbf{Y})}{|\mathbf{n}|!Z_{\mathbf{n}}(\mathbf{I}_n)}. \tag{1.3.2}$$

Here $\mathbf{X},\mathbf{Y}$ are Hermitian $N\times N$ matrices and $Z_{\mathbf{n}}(\mathbf{X}),Z_{\mathbf{n}}(\mathbf{Y})$ are zonal spherical polynomials for the symmetric spaces $GL(N,C)/U(N)$, $GL(N,R)/SO(N)$ and $GL(N,H)/Sp(N)$ see [13]. In our paper we shall consider only the first case; different hypergeometric functions related to zonal polynomials for symmetric spaces $GL(N,C)/U(N)$, $GL(N,R)/SO(N)$ and $GL(N,H)/Sp(N)$ [13] will not be considered.

There are also hypergeometric functions related to Jack polynomials $C_{\mathbf{n}}^{(d)}$ [13]:

$$\begin{aligned}&{}_p\mathcal{F}_s^{(d)}\left(a_1,\dots,a_p;b_1,\dots b_s;\mathbf{x}_{(N)},\mathbf{y}_{(N)}\right)=\\ &\sum_{\mathbf{n}}\frac{(a_1)_{\mathbf{n}}^{(d)}\cdots(a_p)_{\mathbf{n}}^{(d)}}{(b_1)_{\mathbf{n}}^{(d)}\cdots(b_s)_{\mathbf{n}}^{(d)}}\frac{C_{\mathbf{n}}^{(d)}(\mathbf{x}_{(N)})C_{\mathbf{n}}^{(d)}(\mathbf{y}_{(N)})}{|\mathbf{n}|!C_{\mathbf{n}}^{(d)}(1^n)},\end{aligned} \tag{1.3.3}$$

where

$$(a)_{\mathbf{n}}^{(d)}=\prod_{i=1}^{l(\mathbf{n})}\left(a-\frac{d}{2}(i-1)\right)_{n_i}. \tag{1.3.4}$$

Here $(c)_k=c(c+1)\cdots(c+k-1)$. It is known that for the special value $d=2$ the last expression (1.3.3) coincides with (1.2.1), and coincides with (1.3.2) as $|q|\to 1$. These last cases we shall consider below.

# 2 A brief introduction to the fermionic description of the KP and TL hierarchies [2, 10, 15]

## 2.1 Fermionic operators and Fock space

We have fermionic fields:

$$\psi(z)=\sum_k\psi_k z^k,\qquad \psi^*(z)=\sum_k\psi_k^* z^{-k-1}dz, \tag{2.1.1}$$

where fermionic operators satisfy the canonical anti-commutation relations:

$$[\psi_m,\psi_n]_+=[\psi_m^*,\psi_n^*]_+=0;\qquad [\psi_m,\psi_n^*]_+=\delta_{mn}. \tag{2.1.2}$$

Let us introduce left and right vacuums by the properties:

$$\psi_m|0\rangle=0\qquad(m<0),\qquad \psi_m^*|0\rangle=0\qquad(m\ge 0), \tag{2.1.3}$$
$$\langle 0|\psi_m=0\qquad(m\ge 0),\qquad \langle 0|\psi_m^*=0\qquad(m<0). \tag{2.1.4}$$

The vacuum expectation value is defined by relations:

$$\langle 0|1|0\rangle=1,\quad \langle 0|\psi_m\psi_m^*|0\rangle=1\quad m<0,\quad \langle 0|\psi_m^*\psi_m|0\rangle=1\quad m\ge 0, \tag{2.1.5}$$

$$\langle 0|\psi_m\psi_n|0\rangle=\langle 0|\psi_m^*\psi_n^*|0\rangle=0,\quad \langle 0|\psi_m\psi_n^*|0\rangle=0\quad m\ne n. \tag{2.1.6}$$

Let us notice that relations (2.1.2)-(2.1.6) are invariant under the transformation

$$\psi_n\to e^{-T_n}\psi_n,\qquad \psi_n^*\to e^{T_n}\psi_n^*\qquad(T_n\in C). \tag{2.1.7}$$

Consider infinite matrices $(a_{ij})_{i,j\in Z}$ satisfying the condition: there exists an $N$ such that $a_{ij} = 0$ for $|i-j| > N$. Let us take the set of linear combinations of quadratic elements $\sum a_{ij} : \psi_i\psi_j^* :$, where :: means the normal ordering $: \psi_i\psi_j^* := \psi_i\psi_j^* - \langle 0|\psi_i\psi_j^*|0\rangle$. These elements together with 1 span an infinite dimensional Lie algebra $\widehat{gl}(\infty)$:

$$[\sum a_{ij} : \psi_i\psi_j^* :, \sum b_{ij} : \psi_i\psi_j^* :] = \sum c_{ij} : \psi_i\psi_j^* : +c_0, \tag{2.1.8}$$

$$c_{ij} = \sum_k a_{ik}b_{kj} - \sum_k b_{ik}a_{kj}, \tag{2.1.9}$$

$$c_0 = \sum_{i<0,j\geq 0} a_{ij}b_{ji} - \sum_{i\geq 0,j<0} a_{ij}b_{ji}. \tag{2.1.10}$$

Now we define the operator $g$ which is an element of the group corresponding to the Lie algebra $\widehat{gl}(\infty)$:

$$g\psi_n g^{-1} = \sum_m \psi_m a_{mn}, \qquad g^{-1}\psi_n^* g = \sum_m a_{nm}\psi_m^*. \tag{2.1.11}$$

## 2.2 The KP and Toda tau functions

First let us define the vacuum vectors labeled by the integer $M$:

$$\langle M| = \langle 0|\Psi_M^*, \qquad |M\rangle = \Psi_M|0\rangle, \tag{2.2.1}$$

$$\begin{aligned} \Psi_M &= \psi_{M-1}\cdots\psi_1\psi_0 \quad M>0, \qquad & \Psi_M &= \psi_M^*\cdots\psi_{-2}^*\psi_{-1}^* \quad M<0, \\ \Psi_M^* &= \psi_0^*\psi_1^*\cdots\psi_{M-1}^* \quad M>0, \qquad & \Psi_M^* &= \psi_{-1}\psi_{-2}\cdots\psi_M \quad M<0. \end{aligned} \tag{2.2.2}$$

The tau-function of the KP equation and the tau-function of the two-dimensional Toda lattice (TL) sometimes are defined as

$$\tau_{KP}(M,\mathbf{t}) = \langle M|e^{H(\mathbf{t})}g|M\rangle, \tag{2.2.3}$$

$$\tau_{TL}(M,\mathbf{t},\mathbf{t}^*) = \langle M|e^{H(\mathbf{t})}ge^{H^*(\mathbf{t}^*)}|M\rangle. \tag{2.2.4}$$

According to [10] the integer $M$ in (2.2.4) plays the role of discrete Toda lattice variable.

The times $\mathbf{t} = (t_1, t_2, \ldots)$ and $\mathbf{t}^* = (t_1^*, t_2^*, \ldots)$ are called higher Toda lattice times [10, 15] (the first set $\mathbf{t}$ is in the same time the set of higher KP times. The first times of this set $t_1, t_2, t_3$ are independent variables for KP equation (0.0.1), which is the first nontrivial equation in the KP hierarchy). $H(\mathbf{t})$ and $H^*(\mathbf{t}^*)$ belong to the following $\widehat{gl}(\infty)$ Cartan subalgebras:

$$H(\mathbf{t}) = \sum_{n=1}^{+\infty} t_n H_n, \quad H^*(\mathbf{t}^*) = \sum_{n=1}^{+\infty} t_n^* H_{-n}, \quad H_n = \frac{1}{2\pi i}\oint : z^n\psi(z)\psi^*(z) : . \tag{2.2.5}$$

For the Hamiltonians we have Heisenberg algebra commutation relations:

$$[H_n, H_m] = n\delta_{m+n,0}. \tag{2.2.6}$$

The action of $e^{H(\mathbf{t})}$ on the fermions:

$$e^{H(\mathbf{t})}\psi_i e^{-H(\mathbf{t})} = \sum_{n=0}^{+\infty} p_n(\mathbf{t})\psi_{i-n}, \quad e^{H(\mathbf{t})}\psi_i^* e^{-H(\mathbf{t})} = \sum_{n=0}^{+\infty} p_n(\mathbf{t})\psi_{i+n}^*, \tag{2.2.7}$$

where $p_n$ is the elementary Schur polynomial defined by the Taylor's expansion:

$$e^{\xi(\mathbf{t},z)} = \exp(\sum_{k=1}^{+\infty} t_k z^k) = \sum_{n=0}^{+\infty} z^n p_n(\mathbf{t}). \tag{2.2.8}$$

The action on fermionic fields is especially simple:

$$e^{H(\mathbf{t})}\psi(z)e^{-H(\mathbf{t})}=\psi(z)e^{\xi(\mathbf{t},z)},\quad e^{H(\mathbf{t})}\psi^*(z)e^{-H(\mathbf{t})}=\psi^*(z)e^{-\xi(\mathbf{t},z)},\tag{2.2.9}$$

$$e^{-H^*(\mathbf{t}^*)}\psi(z)e^{H^*(\mathbf{t}^*)}=\psi(z)e^{-\xi(\mathbf{t}^*,z^{-1})},\quad e^{-H^*(\mathbf{t}^*)}\psi^*(z)e^{H^*(\mathbf{t}^*)}=\psi^*(z)e^{\xi(\mathbf{t}^*,z^{-1})}.\tag{2.2.10}$$

In the KP theory it is suitable to use another definition of the Schur function corresponding to the partition $\mathbf{n}=(n_1,\dots,n_r)$:

$$s_{\mathbf{n}}(\mathbf{t})=\det(p_{n_i-i+j}(\mathbf{t}))_{1\le i,j\le r},\tag{2.2.11}$$

where $p_m(\mathbf{t})$ is the elementary Schur polynomial defined by the Taylor's expansion:

$$e^{\xi(\mathbf{t},z)}=\exp(\sum_{k=1}^{+\infty}t_kz^k)=\sum_{n=0}^{+\infty}z^np_n(\mathbf{t}).\tag{2.2.12}$$

It is related to $s_{\mathbf{n}}\left(\mathbf{x}_{(N)}\right)$ and $s_{\mathbf{n}'}\left(\mathbf{x}_{(N)}\right)$, where a partition $\mathbf{n}'$ is conjugated to $\mathbf{n}$, as follows [26]:

$$s_{\mathbf{n}}(\mathbf{t}^+(\mathbf{x}_{(N)}))=s_{\mathbf{n}}\left(\mathbf{x}_{(N)}\right),\quad s_{\mathbf{n}}(\mathbf{t}^-(\mathbf{x}_{(N)}))=s_{\mathbf{n}'}\left(\mathbf{x}_{(N)}\right)\tag{2.2.13}$$

via the changes of variables (which is known as Miwa change of variables in the literature on the integrable systems):

$$t_m^+(\mathbf{x}_{(N)})=\sum_{i=1}^{N}\frac{x_i^m}{m},\tag{2.2.14}$$

$$t_m^-(\mathbf{x}_{(N)})=-\sum_{i=1}^{N}\frac{x_i^m}{m}.\tag{2.2.15}$$

Let us notice that $s_{\mathbf{n}}(\mathbf{t}^+(\mathbf{x}_{(N)}))=0$ for $l(\mathbf{n})>N$, and $s_{\mathbf{n}}(\mathbf{t}^-(\mathbf{x}_{(N)}))=0$ for $l(\mathbf{n}')>N$.

**Lemma 1** [2]

For $-j_1<\cdots<-j_k<0\le i_s<\cdots<i_1$, $s-k\ge 0$ the next formula is valid:

$$\langle s-k|e^{H(\mathbf{t})}\psi^*_{-j_1}\cdots\psi^*_{-j_k}\psi_{i_s}\cdots\psi_{i_1}|0\rangle=(-1)^{j_1+\cdots+j_k+(k-s)(k-s+1)/2}s_{\mathbf{n}}(\mathbf{t}),\tag{2.2.16}$$

where the partition $\mathbf{n}=(n_1,\dots,n_{s-k},n_{s-k+1},\dots,n_{s-k+j_1})$ is defined by the pair of partitions:

$$(n_1,\dots,n_{s-k})=(i_1-(s-k)+1,i_2-(s-k)+2,\dots,i_{s-k}),\tag{2.2.17}$$

$$(n_{s-k+1},\dots,n_{s-k+j_1})=(i_{s-k+1},\dots,i_s|j_1-1,\dots,j_k-1).\tag{2.2.18}$$

The proof is achieved by direct calculation. Here $(\dots|\dots)$ is another notation for a partition due to Frobenius (see [26]).

## 2.3 Baker-Akhiezer functions and bilinear identities

Vertex operators $V_\infty(z)$, $V_\infty^*(z)$ and $V_0(z)$, $V_0^*(z)$ act on the space $C[t_1,t_2,\dots]$ of polynomials in infinitely many variables, and are defined by the formulae:

$$V_\infty(z)=z^Me^{\xi(\mathbf{t},z)}e^{-\xi(\tilde{\partial},z^{-1})},\quad V_\infty^*(z)=z^{-M}e^{-\xi(\mathbf{t},z)}e^{\xi(\tilde{\partial},z^{-1})},\tag{2.3.1}$$

$$V_0(z)=z^{-M}e^{\xi(\mathbf{t}^*,z^{-1})}e^{-\xi(\tilde{\partial}^*,z)},\quad V_0^*(z)=z^Me^{-\xi(\mathbf{t}^*,z^{-1})}e^{\xi(\tilde{\partial}^*,z)},\tag{2.3.2}$$

where $\tilde{\partial}=(\frac{\partial}{\partial t_1},\frac{1}{2}\frac{\partial}{\partial t_2},\frac{1}{3}\frac{\partial}{\partial t_3},\dots)$, $\tilde{\partial}^*=(\frac{\partial}{\partial t_1^*},\frac{1}{2}\frac{\partial}{\partial t_2^*},\frac{1}{3}\frac{\partial}{\partial t_3^*},\dots)$.

We have the rules of the bosonization:

$$\langle M+1|e^{H(\mathbf{t})}\psi(z)=V_\infty(z)\langle M|e^{H(\mathbf{t})},\quad \langle M-1|e^{H(\mathbf{t})}\psi^*(z)=V_\infty^*(z)\langle M|e^{H(\mathbf{t})}, \tag{2.3.3}$$

$$\psi^*(z)e^{H^*(\mathbf{t}^*)}|M\rangle=V_0^*(z)e^{H^*(\mathbf{t}^*)}|M+1\rangle,\quad \psi(z)e^{H^*(\mathbf{t}^*)}|M\rangle=V_0(z)e^{H^*(\mathbf{t}^*)}|M-1\rangle. \tag{2.3.4}$$

The Baker-Akhiezer functions and conjugated Baker-Akhiezer functions are:

$$w_\infty(M,\mathbf{t},\mathbf{t}^*,z)=\frac{V_\infty(z)\tau}{\tau},\quad w_\infty^*(M,\mathbf{t},\mathbf{t}^*,z)=\frac{V_\infty^*(z)\tau}{\tau}, \tag{2.3.5}$$

$$w_0(M,\mathbf{t},\mathbf{t}^*,z)=\frac{V_0(z)\tau(M+1)}{\tau(M)},\quad w_0^*(M,\mathbf{t},\mathbf{t}^*,z)=\frac{V_0^*(z)\tau(M-1)}{\tau(M)}, \tag{2.3.6}$$

where

$$\tau(M,\mathbf{t},\mathbf{t}^*)=\langle M|e^{H(\mathbf{t})}ge^{H^*(\mathbf{t}^*)}|M\rangle. \tag{2.3.7}$$

Both KP and TL hierarchies are described by the bilinear identity:

$$\oint w_\infty(M,\mathbf{t},\mathbf{t}^*,z)w_\infty^*(M',\mathbf{t}',\mathbf{t}'^*,z)dz=\oint w_0(M,\mathbf{t},\mathbf{t}^*,z^{-1})w_0^*(M',\mathbf{t}',\mathbf{t}'^*,z^{-1})z^{-2}dz, \tag{2.3.8}$$

which holds for any $\mathbf{t},\mathbf{t}^*,\mathbf{t}',\mathbf{t}'^*$ for any integers $M,M'$.

The Schur functions $s_{\mathbf{n}}(\mathbf{t})$ are well-known examples of tau-functions which correspond to rational solutions of the KP hierarchy. It is known that not any linear combination of Schur functions turns to be a KP tau-function, in order to find these combinations one should solve bilinear difference equation, see [10], which is actually a version of discrete Hirota equation. Below we shall present KP tau-functions which are infinite series of Schur polynomials, and which turn to be known hypergeometric functions (1.2.1),(1.3.2). We shall use the fermionic representation of tau-function [10].

# 3 Hypergeometric functions related to Schur functions

## 3.1 KP tau-function $\tau_r(M,\mathbf{t},\beta)$

Let $r$ be a function of one variable. Let $D=z\frac{d}{dz}$ acts on the basis $\{z^n;n\in Z\}$ of functions holomorphic in the punctured disk $0<|z|<1$ . Then we put $r(D)z^n=r(n)z^n$. All functions of operator $D$ which we consider below are given via their eigenvalues on this basis.

Let us consider an abelian subalgebra in $\widehat{gl}(\infty)$ formed by the set of fermionic operators

$$A_k=\frac{1}{2\pi i}\oint\psi^*(z)\left(\frac{1}{z}r(D)\right)^k\psi(z),\quad k=1,2,\ldots, \tag{3.1.1}$$

where the operator $r(D)$ acts on all functions of $z$ from the right hand side. In other terms

$$A_k=\sum_{n=-\infty}^{\infty}\psi_{n-k}^*\psi_n r(n)r(n-1)\cdots r(n-k+1),\quad k=1,2,\ldots. \tag{3.1.2}$$

We have $[A_m,A_k]=0$ for each $m,k$. Fermionic operators (3.1.2) resemble Toda lattice Hamiltonians $-H_k^*$ (2.2.5), and coincide with them if $r(n)=1,n\in Z$.

For the collection of independent variables $\beta = (\beta_1, \beta_2, \ldots)$ we denote

$$A(\beta) = \sum_{n=1}^{\infty} \beta_n A_n. \tag{3.1.3}$$

For the partition $\mathbf{n} = (n_1, \ldots, n_k)$ and a function of one variable $r$, let us introduce the notation

$$r_{\mathbf{n}}(M) = \prod_{i=1}^{k} r(1-i+M)r(2-i+M)\cdots r(n_i-i+M). \tag{3.1.4}$$

We set $r_{\mathbf{0}}(M) = 1$. Using the notation from (2.2.16) we have
**Lemma 2** The following formula holds

$$\langle 0|\psi^*_{i_1}\cdots\psi^*_{i_s}\psi_{-j_s}\cdots\psi_{-j_1}e^{-A(\beta)}|0\rangle = (-1)^{j_1+\cdots+j_s} r_{\mathbf{n}}(0)s_{\mathbf{n}}(\beta). \tag{3.1.5}$$

The proof is achieved by a direct calculation using $e^A = 1 + A + \frac{1}{2}A^2 + \cdots$, (3.1.2), the hook decomposition of $\mathbf{n}$ and (2.2.16).
Let us consider the tau-function (2.2.3) of the KP hierarchy

$$\tau_r(M, \mathbf{t}, \beta) := \langle M|e^{H(\mathbf{t})}e^{-A(\beta)}|M\rangle. \tag{3.1.6}$$

Using Taylor expanding $e^H = 1 + H + \cdots$ and *Lemma 1*, *Lemma 2* we easily get

**Proposition 1** *We have the expansion:*

$$\tau_r(M, \mathbf{t}, \beta) = \sum_{\mathbf{n}} r_{\mathbf{n}}(M)s_{\mathbf{n}}(\mathbf{t})s_{\mathbf{n}}(\beta). \tag{3.1.7}$$

We shall not consider the problem of convergence of this series. The variables $M, \mathbf{t}$ play the role of KP higher times, $\beta$ is a collection of group times for a commuting subalgebra of additional symmetries of KP (see [24, 22, 23] and Remark 7 in [25]). From different point of view (3.1.7) is a tau-function of two-dimensional Toda lattice [15] with two sets of continuous variables $\mathbf{t}$, $\beta$ and one discrete variable $M$. Formula (3.1.7) is symmetric with respect to $\mathbf{t} \leftrightarrow \beta$. This 'duality' supplies us with the string equations [21] which characterize a tau-function of hypergeometric type (see below). In [16] the similar expansions to (3.1.7) were considered, without specifying the coefficients and in a different context.
For given $r$ we define the function $r'$:

$$r'(n) := r(-n). \tag{3.1.8}$$

**Proposition 2** *We have the involution:*

$$\tau_{r'}(-M, -\mathbf{t}, -\beta) = \tau_r(M, \mathbf{t}, \beta). \tag{3.1.9}$$

The proof follows from the relations

$$r'_{\mathbf{n}}(M) = r_{\mathbf{n}'}(-M), \quad s_{\mathbf{n}}(\mathbf{t}) = s_{\mathbf{n}'}(-\mathbf{t}). \tag{3.1.10}$$

Now let us introduce

$$\tilde{A}_k = -\frac{1}{2\pi i}\oint \psi^*(z)\left(\tilde{r}(D)z\right)^k\psi(z), \quad (k = 1, 2, \ldots), \quad \tilde{A}(\beta) = \sum_{n=1}^{\infty}\tilde{\beta}_n\tilde{A}_n. \tag{3.1.11}$$

Then we have the following generalization of Proposition 1:

**Proposition 3**

$$\langle M|e^{\tilde{A}(\tilde{\beta})}e^{-A(\beta)}|M\rangle = \sum_{\mathbf{n}} (\tilde{r}r)_{\mathbf{n}}(M)s_{\mathbf{n}}(\tilde{\beta})s_{\mathbf{n}}(\beta). \tag{3.1.12}$$

**Remark 1** *This expansion has the following interpretation. If one uses vacuum vectors* $\langle M||M\rangle$ *for normal ordering* $: A := A - \langle M|A|M\rangle$ *in the formula for* $\widehat{gl}(\infty)$ *commutation relation (2.1.8), he gets different* $\widehat{gl}(\infty)$ *2-cocycles* $c_M$ *which are cohomological to* $c_0$ *(2.1.10). The value of* $c_M$ *on the elements* $\tilde{A}_1, A_1$ *is* $\tilde{r}(M)r(M)$*:*

$$c_M(\tilde{A}_1, A_1) = (\tilde{r}r)(M). \tag{3.1.13}$$

*Formula (3.1.12) is an expansion of* $\widehat{GL}(\infty)$ *group 2-cocycle, evaluated on the elements* $e^{\tilde{A}(\tilde{\beta})}$*,*$e^{-A(\beta)}$*, in terms of* $r(M)$*, see also* Appendix "Gauss factorization problem etc."

In what follows we put $\tilde{r} = 1$, since (3.1.12) depends only on $\tilde{r}r$.

## 3.2 $H_0(\mathbf{T})$, twisted fermions $\psi(\mathbf{T}, z), \psi^*(\mathbf{T}, z)$ and bosonization rules

Let $r \neq 0$, and put $r(n) = e^{T_{n-1}-T_n}$, where the variables $T_n$ are defined up to a constant independent of $n$. We define a Hamiltonian $H_0(\mathbf{T}) \in \widehat{gl}(\infty)$ (all $T_n \in C$ are finite):

$$H_0(\mathbf{T}) := \sum_{n=-\infty}^{\infty} T_n : \psi_n^* \psi_n :, \tag{3.2.1}$$

which produces the transformation (2.1.7):

$$e^{\mp H_0(\mathbf{T})}\psi_n e^{\pm H_0(\mathbf{T})} = e^{\pm T_n}\psi_n, \quad e^{\mp H_0(\mathbf{T})}\psi_n^* e^{\pm H_0(\mathbf{T})} = e^{\mp T_n}\psi_n^*, \tag{3.2.2}$$

$$e^{H_0(\mathbf{T})}\tilde{A}(\tilde{\beta})e^{-H_0(\mathbf{T})} = H(\tilde{\beta}), \quad e^{-H_0(\mathbf{T})}A(\beta)e^{H_0(\mathbf{T})} = -H^*(\beta). \tag{3.2.3}$$

Let $r \neq 0$. It is convenient to consider the fermionic operators:

$$\psi(\mathbf{T}, z) = e^{H_0(\mathbf{T})}\psi(z)e^{-H_0(\mathbf{T})} = \sum_{n=-\infty}^{n=+\infty} e^{-T_n} z^n \psi_n, \tag{3.2.4}$$

$$\psi^*(\mathbf{T}, z) = e^{H_0(\mathbf{T})}\psi^*(z)e^{-H_0(\mathbf{T})} = \sum_{n=-\infty}^{n=+\infty} e^{T_n} z^{-n} \psi_n^* \frac{dz}{z}. \tag{3.2.5}$$

For the variables $\mathbf{t}^+(\mathbf{x}_{(N)})$ and $\mathbf{t}^{*+}(\mathbf{y}_{(N)})$, and for the "Hamiltonians" $A$ and $\tilde{A}$ defined by (3.1.1), (3.1.11), one can derive the bosonization rules:

$$e^{-A(\mathbf{t}^{*+}(\mathbf{y}_{(N)}))}|M\rangle = \frac{\psi(\mathbf{T}, y_1)\cdots\psi(\mathbf{T}, y_N)|M-N\rangle}{\Delta^+(M, N, \mathbf{T}, \mathbf{y}_{(N)})}, \tag{3.2.6}$$

$$e^{-A(\mathbf{t}^{*-}(\mathbf{y}_{(N)}))}|M\rangle = \frac{\psi^*(\mathbf{T}, y_1)\cdots\psi^*(\mathbf{T}, y_N)|M+N\rangle}{\Delta^-(M, N, \mathbf{T}, \mathbf{y}_{(N)})}, \tag{3.2.7}$$

$$\langle M|e^{\tilde{A}(\mathbf{t}^+(\mathbf{x}_{(N)}))} = \frac{\langle M-N|\psi^*(-\tilde{\mathbf{T}}, \frac{1}{x_N})\cdots\psi^*(-\tilde{\mathbf{T}}, \frac{1}{x_1})}{\tilde{\Delta}^+(M, N, \tilde{\mathbf{T}}, \mathbf{x}_{(N)})}, \tag{3.2.8}$$

$$\langle M|e^{\tilde{A}(\mathbf{t}^-(\mathbf{x}_{(N)}))} = \frac{\langle M+N|\psi(-\tilde{\mathbf{T}}, \frac{1}{x_N})\cdots\psi(-\tilde{\mathbf{T}}, \frac{1}{x_1})}{\tilde{\Delta}^-(M, N, \tilde{\mathbf{T}}, \mathbf{x}_{(N)})}. \tag{3.2.9}$$

Here $\tilde{T}_n$ are related to $\tilde{A}$ via (3.1.11) and $\tilde{r}(n) = e^{\tilde{T}_{n-1}-\tilde{T}_n}$. Vandermond coefficients are

$$\Delta^+(M,N,\mathbf{T},\mathbf{y}_{(N)}) = \frac{\prod_{i<j}(y_i-y_j)}{(y_1\cdots y_N)^{N-M}}\frac{\tau(M,\mathbf{0},\mathbf{T},\mathbf{0})}{\tau(M-N,\mathbf{0},\mathbf{T},\mathbf{0})}, \tag{3.2.10}$$

$$\Delta^-(M,N,\mathbf{T},\mathbf{y}_{(N)}) = \frac{\prod_{i<j}(y_i-y_j)}{(y_1\cdots y_N)^{M+N}}\frac{\tau(M,\mathbf{0},\mathbf{T},\mathbf{0})}{\tau(M+N,\mathbf{0},\mathbf{T},\mathbf{0})}, \tag{3.2.11}$$

$$\tilde{\Delta}^+(M,N,\tilde{\mathbf{T}},\mathbf{x}_{(N)}) = \frac{\prod_{i<j}(x_i-x_j)}{(x_1\cdots x_N)^{N-M-1}}\frac{\tau(M,\mathbf{0},\tilde{\mathbf{T}},\mathbf{0})}{\tau(M-N,\mathbf{0},\tilde{\mathbf{T}},\mathbf{0})}, \tag{3.2.12}$$

$$\tilde{\Delta}^-(M,N,\tilde{\mathbf{T}},\mathbf{x}_{(N)}) = \frac{\prod_{i<j}(x_i-x_j)}{(x_1\cdots x_N)^{N+M-1}}\frac{\tau(M,\mathbf{0},\tilde{\mathbf{T}},\mathbf{0})}{\tau(M+N,\mathbf{0},\tilde{\mathbf{T}},\mathbf{0})}. \tag{3.2.13}$$

The notation $\tau(M,\mathbf{0},\mathbf{T},\mathbf{0})$ is explained in the next Subsection, see (3.3.5),(3.3.6). Therefore in Miwa variables one can rewrite correlators (3.1.12):

$$\langle M|e^{\tilde{A}(\mathbf{t}^+(\mathbf{x}_{(N)}))}e^{-A(\mathbf{t}^{*+}(\mathbf{y}_{(N)}))}|M\rangle = \frac{\langle M-N|\psi^*(-\tilde{\mathbf{T}},\frac{1}{x_N})\dots\psi^*(-\tilde{\mathbf{T}},\frac{1}{x_1})\psi(\mathbf{T},y_1)\cdots\psi(\mathbf{T},y_N)|M-N\rangle}{\tilde{\Delta}^+(M,N,\tilde{\mathbf{T}},\mathbf{x}_{(N)})\Delta^+(M,N,\mathbf{T},\mathbf{y}_{(N)})}, \tag{3.2.14}$$

$$\langle M|e^{\tilde{A}(\mathbf{t}^-(\mathbf{x}_{(N)}))}e^{-A(\mathbf{t}^{*-}(\mathbf{y}_{(N)}))}|M\rangle = \frac{\langle M+N|\psi(-\tilde{\mathbf{T}},\frac{1}{x_N})\cdots\psi(-\tilde{\mathbf{T}},\frac{1}{x_1})\psi^*(\mathbf{T},y_1)\cdots\psi^*(\mathbf{T},y_N)|M+N\rangle}{\tilde{\Delta}^-(M,N,\tilde{\mathbf{T}},\mathbf{x}_{(N)})\Delta^-(M,N,\mathbf{T},\mathbf{y}_{(N)})}. \tag{3.2.15}$$

## 3.3 Toda lattice tau-function $\tau(M,\mathbf{t},\mathbf{T},\mathbf{t}^*)$

Now let us consider the *Toda lattice* tau-function (2.2.4), which depends on the three sets of variables $\mathbf{t},\mathbf{T},\mathbf{t}^*$ and on $M\in Z$:

$$\tau(M,\mathbf{t},\mathbf{T},\mathbf{t}^*) = \langle M|e^{H(\mathbf{t})}\exp\left(\sum_{-\infty}^{\infty}T_n:\psi_n^*\psi_n:\right)e^{H^*(\mathbf{t}^*)}|M\rangle, \tag{3.3.1}$$

where $:\psi_n^*\psi_n := \psi_n^*\psi_n - \langle 0|\psi_n^*\psi_n|0\rangle$. Since the operator $\sum_{-\infty}^{\infty}:\psi_n^*\psi_n:$ commutes with all elements of the $\widehat{gl}(\infty)$ algebra, one can put $T_{-1}=0$ in (3.3.1). With respect to the KP and the TL dynamics the times $T_n$ have a meaning of integrals of motion. With respect to each pair of times $(t_m,T_n)$ one can consider the Liouville equation related to (3.3.1), see *Appendix "Equations with respect to* $\mathbf{T}$ *variables"* (the variables $\mathbf{t}^*$ plays the role of integrals of motion for these Liouville equations).

As we shall see the hypergeometric functions (1.1.1),(1.1.3),(1.2.1),(1.2.6) listed in the Introduction are ratios of tau-functions (3.3.1) evaluated at special values of times $M,\mathbf{t},\mathbf{T},\mathbf{t}^*$. It is true only in the case when all parameters $a_k$ of the hypergeometric functions are nonintegers. For the case when at least one of the indices $a_k$ is an integer, we will need a tau-function of an open Toda chain which will be considered in the next Sections.

Tau-function (3.3.1) is linear in each $e^{T_n}$. It is described by the Proposition

**Proposition 4**

$$\frac{\tau(M,\mathbf{t},\mathbf{T},\mathbf{t}^*)}{\tau(M,\mathbf{0},\mathbf{T},\mathbf{0})} = 1+\sum_{\mathbf{n}\neq\mathbf{0}}e^{(T_{M-1}-T_{n_1+M-1})+(T_{M-2}-T_{n_2+M-2})+\cdots+(T_{M-l}-T_{n_l+M-l})}s_{\mathbf{n}}(\mathbf{t})s_{\mathbf{n}}(\mathbf{t}^*). \tag{3.3.2}$$

*The sum is going over all different partitions*

$$\mathbf{n} = (n_1, n_2, \dots, n_l), \quad l = 1, 2, 3, ..., \tag{3.3.3}$$

*excluding the partition* $\mathbf{0}$.

Let $r \neq 0$,$\tilde{r} \neq 0$. Then we put

$$r(n) = e^{T_{n-1}-T_n}, \quad \tilde{r}(n) = e^{\tilde{T}_{n-1}-\tilde{T}_n}. \tag{3.3.4}$$

Let us show the equivalence of (3.1.12) and (3.3.2) in this case. We have $\tau(0, \mathbf{0}, \mathbf{T}, \mathbf{0}) = 1$ and

$$\tau(n, \mathbf{0}, \mathbf{T}, \mathbf{0}) = e^{-T_{n-1}-\dots-T_1-T_0}, \quad n > 0, \tag{3.3.5}$$

$$\tau(n, \mathbf{0}, \mathbf{T}, \mathbf{0}) = e^{T_n+\dots+T_{-2}+T_{-1}}, \quad n < 0. \tag{3.3.6}$$

**Proposition 5** *Let* $\tau(n, \tilde{\beta}, \mathbf{T}, \beta)$ *is Toda lattice tau-function (3.3.1) and* $\tau_{\tilde{r}r}(n, \tilde{\beta}, \beta)$ *is defined by (3.1.12), where* $\tilde{r}, r, \mathbf{T}, \tilde{\mathbf{T}}$ *are related by (3.3.4), the functions* $r, \tilde{r}$ *have no zeroes at integer values of argument then*

$$\frac{\tau(n, \tilde{\beta}, \tilde{\mathbf{T}} + \mathbf{T}, \beta)}{\tau(n, \mathbf{0}, \tilde{\mathbf{T}} + \mathbf{T}, \mathbf{0})} = \langle n | e^{\tilde{A}(\tilde{\beta})} e^{-A(\beta)} | n \rangle = \tau_{\tilde{r}r}(n, \tilde{\beta}, \beta). \tag{3.3.7}$$

*This proposition follows from formulas (3.2.2)-(3.2.3).*

For $\tilde{r} = 1$ we can put $\tilde{\beta} = \mathbf{t}$. Then the next equations hold

$$\partial_{t_1} \partial_{\beta_1} \phi_n = r(n) e^{\phi_{n-1}-\phi_n} - r(n+1) e^{\phi_n - \phi_{n+1}}, \quad e^{-\phi_n} = \frac{\tau_r(n+1, \mathbf{t}, \beta)}{\tau_r(n, \mathbf{t}, \beta)}, \tag{3.3.8}$$

$$(\tau(n) := \tau_r(n, \mathbf{t}, \beta)) \qquad \tau(n) \partial_{\beta_1} \partial_{t_1} \tau(n) - \partial_{t_1} \tau(n) \partial_{\beta_1} \tau(n) = r(n) \tau(n-1) \tau(n+1). \tag{3.3.9}$$

As we shall see eqs. (3.3.8) and (3.3.9) are still true in case $r(n)$ has zeroes.
If the function $r$ has no integer zeroes, using the change of variables

$$\varphi_n = -\phi_n - T_n, \tag{3.3.10}$$

we obtain Toda lattice equation in standard form [15]:

$$\partial_{t_1} \partial_{t_1^*} \varphi_n = e^{\varphi_{n+1}-\varphi_n} - e^{\varphi_n - \varphi_{n-1}}. \tag{3.3.11}$$

As we see the variables $T_n$ might have the meaning of asymptotic values of the fields $\phi_n$ for the class of tau-functions (3.3.1) which is characterized by the property $\varphi_n \to 0$ as $t_1 \to 0$.

## 3.4 Toda lattice consisted of open parts

Now we consider tau-function (3.3.1) with the modification of the definition of flows.

Definition. Let us introduce the function $\delta$ which is equal to zero when $r$ is equal to zero and is equal to unity otherwise:

$$\delta(n) = 0 \quad if \quad r(n) = 0, \qquad \delta(n) = 1 \quad if \quad r(n) \neq 0. \tag{3.4.1}$$

Given collection of zeroes $\mathbf{m}$ of $r$:

$$\mathbf{m} = \{M_i \in Z\}, \quad M_{i+1} > M_i, \quad r(M_i) = 0. \tag{3.4.2}$$

we construct Hamiltonians labeled by $\mathbf{m}$:

$$H_{-k}(\mathbf{m}) = \sum_{n=-\infty}^{+\infty} \delta(n)\delta(n-1)\cdots\delta(n-k+1)\psi_n\psi^*_{n-k}, \quad H^*(\mathbf{m};\beta) = \sum H_{-k}(\mathbf{m})\beta_k. \tag{3.4.3}$$

$$H_k(\mathbf{m}) = \sum_{n=-\infty}^{+\infty} \delta(n+1)\delta(n+2)\cdots\delta(n+k)\psi_n\psi^*_{n+k}, \qquad H(\mathbf{m};\mathbf{t}) = \sum H_k(\mathbf{m})t_k. \tag{3.4.4}$$

The tau-function of the open TL we are interested in, see (3.4.9) below, can be written in the three equivalent forms:

$$\tau_{op}(\mathbf{m}; M, \mathbf{t}, \beta) = \langle M|e^{H(\mathbf{t})} \exp\left(\sum_{-\infty}^{\infty} T_n : \psi^*_n\psi_n :\right) e^{H^*(\mathbf{m};\beta)}|M\rangle = \tag{3.4.5}$$

$$\langle M|e^{H(\mathbf{m};\mathbf{t})} \exp\left(\sum_{-\infty}^{\infty} T_n : \psi^*_n\psi_n :\right) e^{H(\beta)}|M\rangle = \langle M|e^{H(\mathbf{m};\mathbf{t})} \exp\left(\sum_{-\infty}^{\infty} T_n : \psi^*_n\psi_n :\right) e^{H(\mathbf{m};\beta)}|M\rangle, \tag{3.4.6}$$

where $T_n \in C$ are some constants (times). This tau-function has the property:

$$M_i \in \mathbf{m} \Rightarrow \tau_{op}(\mathbf{m}; M_i, \mathbf{t}, \beta) = 1 \tag{3.4.7}$$

for all values of times $\mathbf{t}, \beta$. If one consider

$$\varphi_n = \log \frac{\tau_{op}(\mathbf{m}; n+1, \mathbf{t}, \mathbf{T}, \beta)}{\tau_{op}(\mathbf{m}; n, \mathbf{t}, \mathbf{T}, \beta)} \tag{3.4.8}$$

he comes to the equation of an open TL:

$$\partial_{t_1}\partial_{\beta_1}\varphi_n = \delta(n)e^{\varphi_{n-1}-\varphi_n} - \delta(n+1)e^{\varphi_n-\varphi_{n+1}}. \tag{3.4.9}$$

The set of fields $\varphi_n$ solves a number of open lattice problems in the set of intervals:

$$\{\varphi_n, \quad n < M_1\}, \tag{3.4.10}$$

$$\{\varphi_n, \quad M_i \le n < M_{i+1}, \quad M_{i+1} - M_i > 1\}, \tag{3.4.11}$$

$$\{\varphi_n, \quad n > M_s\}. \tag{3.4.12}$$

The tau-function describes a set of open Toda lattices between each pair of neighbor zeroes (between neighbor zeroes $M_{i+1}$,$M_i$ there is an open chain with $M_{i+1} - M_i$ number of sites), and two semiinfinite Toda lattices, one of them ends on the smallest zero and the other on the largest zero.

## 3.5 Properties of the tau function $\tau_r$ when function r(n) has zeroes

Now let us introduce a set of $T_n$ variables with the help of relations

$$r(n) = e^{T_{n-1}-T_n} \tag{3.5.1}$$

for all $n$ where $r(n) \neq 0$. Equation (3.5.1) define variables $T_n$ uniquely up to an integration constant in each of the intervals (the number of the constants is equal to the number of intervals)

$$\{T_n, \quad n < M_1\}, \tag{3.5.2}$$

$$\{T_n, \quad M_i \le n < M_{i+1}, \quad M_{i+1} - M_i > 1\}, \tag{3.5.3}$$

$$\{T_n, \quad M_s \le n\} \tag{3.5.4}$$

separately. In case of there are zeroes such that $M_{i+1} = M_i + 1$ one can define variables $T_{M_i}$, however we will not need them.

We introduce the Hamiltonian

$$H_0(\mathbf{T}) = \sum_n T_n : \psi_n^* \psi_n :, \tag{3.5.5}$$

where sum is over all $n$ satisfying one of the equations (3.5.2), (3.5.3) or (3.5.4).

We have

$$A_k = -e^{H_0(\mathbf{T})} H_{-k}(\mathbf{m}) e^{-H_0(\mathbf{T})}, \quad A(\beta) = -e^{H_0(\mathbf{T})} H^*(\mathbf{m}, \beta) e^{-H_0(\mathbf{T})}. \tag{3.5.6}$$

**Proposition 6** *Let $\tau_{op}(\mathbf{m}, n, \mathbf{t}, \mathbf{T}, \beta)$ is Toda lattice tau-function (3.4.5), and $\tau_r(n, \mathbf{t}, \beta)$ is defined by (3.1.6), where $r$ and $\mathbf{T}$ are related by (3.5.1), the functions $r$ has zeroes at integer values of argument described by (3.4.2) and for $r \neq 0$ the set of variables $\mathbf{T}$ is related to $r$ by (3.5.1)*

$$\frac{\tau_{op}(\mathbf{m}, n, \mathbf{t}, \mathbf{T}, \beta)}{\tau_{op}(\mathbf{m}, n, \mathbf{0}, \mathbf{T}, \mathbf{0})} = \langle n | e^{H(\mathbf{t})} e^{-A(\beta)} | n \rangle = \tau_r(n, \mathbf{t}, \beta). \tag{3.5.7}$$

*Equations (3.3.8) and (3.3.9) are still true in case $r(n)$ has zeroes.*

Hirota equation (3.3.9) can be viewed as recurrent relation which expresses tau-function with discrete Toda lattice variable $n$ via $\tau_r(M_i \pm 1, \mathbf{t}, \beta)$, $\tau_r(M_i, \mathbf{t}, \beta) = 1$. It follows from (3.1.4),(3.1.7) that

$$r(M_k) = 0 \Rightarrow \tau_r(M_k, \mathbf{t}, \beta) = 1. \tag{3.5.8}$$

Then from (3.1.4),(3.1.7) we see the following. In the region (3.4.11) the series (3.1.7) has only a finite number of nonvanishing terms. For the region (3.4.12) the sum is only over the Young diagrams $\mathbf{n}$ of the length $l(\mathbf{n}) < M - M_1$. For the region (3.4.10) only those diagrams $\mathbf{n}$ for which the conjugated diagrams $\mathbf{n}'$ have length $l(\mathbf{n}') \le M_s - M$ contribute the series (3.1.7).

In *Appendix* we shall write down a system of orthogonal polynomials related to $\mathbf{m}$.

**Remark 2** *There are two different ways to restrict the sum (3.1.7) to a sum over partitions of length $l(\mathbf{n}) \le N$ (or over $l(\mathbf{n}') \le N$). The second way is to use so-called Miwa's change of variables.*

## 3.6 Notations

In order to simplify notations, we shall omit additional argument $\mathbf{m}$ and subindex which distinguish TL tau-function (3.3.1) and open TL tau-functions (3.4.5). Instead of $\tau_{op}(\mathbf{m}, n, \mathbf{t}, \mathbf{T}, \mathbf{t}^*)$ we shall write $\tau(n, \mathbf{t}, \mathbf{T}, \mathbf{t}^*)$. The notation $\tau_r(M, \mathbf{t}, \beta)$ will be used only for the KP tau-function (3.1.6). Also $\beta = \mathbf{t}^*$. When TL higher times are expressed via Miwa change (2.2.14) or (2.2.15), sometimes we shall put the argument $\mathbf{x}_{(N)}$ at the place of the argument $\mathbf{t}$ and the argument $\mathbf{y}_{(N)}$ at the place of $\mathbf{t}^*$, for instance $\tau_r(M, \mathbf{x}_{(N)}, \mathbf{t}^*)$, $\tau_r(M, \mathbf{t}, \mathbf{y}_{(N)})$, $\tau_r(M, \mathbf{x}_{(N)}, \mathbf{y}_{(N)})$.

## 3.7 Linear equations for the tau-function $\tau_r$

Here we shall write down linear equations, which follow from the explicit fermionic representation of the tau-function (3.1.6) via the bosonization formulae (3.2.14) and .(3.2.15) These equations may be also viewed as the constraints which result in the string equations. For the variables $\mathbf{t}^-(\mathbf{x}_{(N)})$, using $\langle M|A=0$ and making profit of the relation $A_k = e^{H_0}H_{-k}e^{-H_0}$ inside the fermionic correlator (3.2.14), we get the partial differential equations for the tau-function (3.1.7):

$$\frac{\partial\tau_r(M,\mathbf{t}^-(\mathbf{x}_{(N)}),\mathbf{t}^*)}{\partial t_k^*} = \frac{1}{\tilde{\Delta}}\left(\sum_{i=1}^{N}(x_i r(-D_{x_i}))^k\right)\tilde{\Delta}\tau_r(M,\mathbf{t}^-(\mathbf{x}_{(N)}),\mathbf{t}^*), \tag{3.7.1}$$

where $\tilde{\Delta}=\tilde{\Delta}^-(M,N,\mathbf{0},\mathbf{x}_{(N)})$. These equations have the meaning of string constraint equations for the tau-function (3.1.7). In variables $\mathbf{t}^{*-}(\mathbf{y}_{(\infty)})$ we can rewrite (3.7.1):

$$(-1)^k\sum_{i=1}^{+\infty}\frac{e_{k-1}\left(\frac{1}{y_1},\dots,\frac{1}{y_{i-1}},\frac{1}{y_{i+1}},\dots\right)}{\prod_{j\neq i}(1-\frac{y_i}{y_j})}\frac{\partial\tau_r(M,\mathbf{t}^-(\mathbf{x}_{(N)}),\mathbf{t}^{*-}(\mathbf{y}_{(\infty)}))}{\partial y_i} =$$
$$\frac{1}{\tilde{\Delta}}\left(\sum_{i=1}^{N}(x_i r(-D_{x_i}))^k\right)\tilde{\Delta}\tau_r(M,\mathbf{t}^-(\mathbf{x}_{(N)}),\mathbf{t}^{*-}(\mathbf{y}_{(\infty)})), \tag{3.7.2}$$

where $e_k(\mathbf{y})$ is a symmetric function defined through the relation $\prod_{i=1}^{+\infty}(1+ty_i)=\sum_{k=0}^{+\infty}t^k e_k(\mathbf{y})$. Also we have

$$\left(\sum_{k=1}^{M+N-1}k-\sum_{i=1}^{N}D_{x_i}\right)\tilde{\Delta}^-\tau(M,\mathbf{t}^-(\mathbf{x}_{(N)}),\mathbf{T},\mathbf{t}^{*-}(\mathbf{y}_{(N')}))\Delta^- =$$
$$\left(\sum_{k=1}^{M+N'-1}k-\sum_{i=1}^{N'}\left(\frac{1}{y_i}D_{y_i}y_i\right)\right)\tilde{\Delta}^-\tau(M,\mathbf{t}^-(\mathbf{x}_{(N)}),\mathbf{T},\mathbf{t}^{*-}(\mathbf{y}_{(N')}))\Delta^-, \tag{3.7.3}$$

where $\tilde{\Delta}^-=\tilde{\Delta}^-(M,N,\mathbf{0},\mathbf{x}_{(N)})$ and $\Delta^-=\Delta^-(M,N,\mathbf{0},\mathbf{y}_{(N')})$. This formula is obtained by the insertion of the fermionic operator $res_z:\psi^*(z)z\frac{d}{dz}\psi(z):$ inside the fermionic correlator. These formulae can be also written in terms of higher KP and TL times, with the help of vertex operator action, see the *Appendix "Vertex operator action"*. Then the relation (3.7.1) is the infinitesimal version of (A3.6), while the relation (3.7.3) is the infinitesimal version of (A3.7).

## 3.8 Determinant formulae

With the help of Wick theorem [10] one obtains the formulae.

**Proposition 7** *A generalization of Milne's determinant formula*

$$\tau_r(M,\mathbf{t}^+(\mathbf{x}_{(N)}),\beta) = \frac{\det\left(x_i^{N-k}\tau_r(M-k+1,\mathbf{t}^+(x_i),\beta)\right)_{i,k=1}^N}{\det\left(x_i^{N-k}\right)_{i,k=1}^N}. \tag{3.8.1}$$

**Proof**

$$\tau_r(M,\mathbf{t}^+(\mathbf{x}_{(N)}),\beta) = \langle M|e^{H(\mathbf{t}^+(\mathbf{x}_{(N)})}e^{-A(\beta)}|M\rangle = \tag{3.8.2}$$

$$\frac{x_1^{N-M-1}\cdots x_N^{N-M-1}}{\prod_{i<j}(x_i-x_j)}\langle M|\psi_{M-1}\cdots\psi_{M-N}\psi^*(\frac{1}{x_N})\cdots\psi^*(\frac{1}{x_1})e^{-A(\beta)}|M\rangle = \tag{3.8.3}$$

$$\frac{(x_1\cdots x_N)^{N-M-1}}{\prod_{i<j}(x_i-x_j)}\det\left(\langle M|\psi_{M-k}\psi^*(\frac{1}{x_i})e^{-A(\beta)}|M\rangle\right)_{i,k=1}^N = \tag{3.8.4}$$

$$\frac{\det\left(x_i^{N-k}\tau_r(M-k+1,\mathbf{t}^+(x_i),\beta)\right)_{i,k=1}^N}{\det\left(x_i^{N-k}\right)_{i,k=1}^N} \tag{3.8.5}$$

*Last equality follows from:*

$$\langle M|\psi_{M-k}\psi^*(\frac{1}{x_i})e^{-A(\beta)}|M\rangle = \tag{3.8.6}$$

$$= \langle M|\psi_{M-1}\cdots\psi_{M-k+1}\psi_{M-k}\psi^*(\frac{1}{x_i})e^{-A(\beta)}\psi^*_{M-k+1}\cdots\psi^*_{M-1}|M\rangle + \tag{3.8.7}$$

$$+\sum_{j=1}^{k-1} a_j^k(\beta)\langle M|\psi_{M-1}\cdots\psi_{M-k+j}\psi^*(\frac{1}{x_i})e^{-A(\beta)}\psi^*_{M-k+j+1}\cdots\psi^*_{M-1}|M\rangle = \tag{3.8.8}$$

$$= \langle M-k+1|\psi_{M-k}\psi^*(\frac{1}{x_i})e^{-A(\beta)}|M-k+1\rangle + \tag{3.8.9}$$

$$+\sum_{j=1}^{k-1} a_j^k(\beta)\langle M-k+1+j|\psi_{M-k+j}\psi^*(\frac{1}{x_i})e^{-A(\beta)}|M-k+1+j\rangle = \tag{3.8.10}$$

$$= x_i^{M-k+1}\tau_r(M-k+1,\mathbf{t}^+(x_i),\beta) + \sum_{j=1}^{k-1} a_j^k(\beta)x_i^{M-k+1+j}\tau_r(M-k+1+j,\mathbf{t}^+(x_i),\beta) \tag{3.8.11}$$

*Where the functions $a_j^k(\beta)$ must be derived as the results of action of operator $e^{-A(\beta)}$ on the fermions $\psi_{M-1},\dots,\psi_{M-k}$. Thus we have:*

$$x_i^{N-M-1}\langle M|\psi_{M-k}\psi^*(\frac{1}{x_i})e^{-A(\beta)}|M\rangle = \tag{3.8.12}$$

$$= x_i^{N-k}\tau_r(M-k+1,\mathbf{t}^+(x_i),\beta) + \sum_{l=1}^{k-1} a_{k-l}^k(\beta)x_i^{N-l}\tau_r(M-l+1,\mathbf{t}^+(x_i),\beta) \tag{3.8.13}$$

**Proposition 8** *For $r\neq 0$ we take a tau function $\tau_r(M,\mathbf{t}^+(\mathbf{x}_{(N)}),\mathbf{t}^{*+}(\mathbf{y}_{(N)}))$ and apply Wick's theorem. We get the determinant formula:*

$$\tau_r(M,\mathbf{t}^+(\mathbf{x}_{(N)}),\mathbf{t}^{*+}(\mathbf{y}_{(N)})) = \frac{\det\left(F(x_iy_j)\right)_{i,j=1}^N}{\tilde\Delta^+(M,N,\mathbf{0},\mathbf{x}_{(N)})\Delta^+(M,N,\mathbf{T},\mathbf{y}_{(N)})}, \tag{3.8.14}$$

$$F(x_iy_j) = \langle M-N|\psi^*\left(\frac{1}{x_i}\right)\psi\left(\mathbf{T},y_j\right)|M-N\rangle. \tag{3.8.15}$$

## 3.9 Integral representations

For the fermions (3.2.4) we easily get the relations:

$$\int\psi(\mathbf{T},\alpha z)d\mu(\alpha) = \psi\left(\mathbf{T}+\mathbf{T}(\mu),z\right), \quad \int\psi^*\left(-\mathbf{T},\frac{1}{\alpha z}\right)d\tilde\mu(\alpha) = \psi^*\left(-\mathbf{T}-\mathbf{T}(\tilde\mu),\frac{1}{z}\right) \tag{3.9.1}$$

where $\mu,\tilde\mu$ are some integration measures, and shifts of times $T_n$ are defined in terms of the moments:

$$\int\alpha^n d\mu(\alpha) = e^{-T_n(\mu)}, \quad \int\alpha^n d\tilde\mu(\alpha) = e^{-T_n(\tilde\mu)}. \tag{3.9.2}$$

Therefore thanks to the bosonization formulae (3.2.14) we have the relations for the tau-function (below $\mathbf{t}^*$ is defined via (2.2.14))

**Proposition 9** *Integral representation formula holds*

$$\int \tilde{\Delta}_{\tilde{\mathbf{T}}}(\tilde{\alpha}\mathbf{x}_{(N)})\frac{\tau\left(M,\mathbf{t}^{+}(\tilde{\alpha}\mathbf{x}_{(N)}),\mathbf{T}+\tilde{\mathbf{T}},\mathbf{t}^{*+}(\alpha\mathbf{y}_{(N)})\right)}{\tau\left(M,\mathbf{0},\mathbf{T}+\tilde{\mathbf{T}},\mathbf{0}\right)}\Delta_{\mathbf{T}}(\alpha\mathbf{y}_{(N)})\prod_{i=1}^{N}d\tilde{\mu}(\tilde{\alpha}_i)\prod_{i=1}^{N}d\mu(\alpha_i)$$

$$=\tilde{\Delta}_{\tilde{\mathbf{T}}+\tilde{\mathbf{T}}(\tilde{\mu})}(\mathbf{x}_{(N)})\frac{\tau\left(M,\mathbf{t}^{+}(\mathbf{x}_{(N)}),\mathbf{T}+\tilde{\mathbf{T}}+\tilde{\mathbf{T}}(\tilde{\mu})+\mathbf{T}(\mu),\mathbf{t}^{*+}(\mathbf{y}_{(N)})\right)}{\tau\left(M,\mathbf{0},\mathbf{T}+\tilde{\mathbf{T}}+\tilde{\mathbf{T}}(\tilde{\mu})+\mathbf{T}(\mu),\mathbf{0}\right)}\Delta_{\mathbf{T}+\mathbf{T}(\mu)}(\mathbf{y}_{(N)}). \quad (3.9.3)$$

*where* $\Delta_{\mathbf{T}}(\alpha\mathbf{y}_{(N)})=\Delta^{+}(M,N,\mathbf{T},\alpha\mathbf{y}_{(N)})$, $\tilde{\Delta}_{\tilde{\mathbf{T}}}(\tilde{\alpha}\mathbf{x}_{(N)})=\tilde{\Delta}^{+}(M,N,\tilde{\mathbf{T}},\tilde{\alpha}\mathbf{x}_{(N)})$, $\alpha\mathbf{y}_{(N)}=(\alpha_1y_1,\alpha_2y_2,\dots,\alpha_Ny_N)$ *and* $\tilde{\alpha}\mathbf{x}_{(N)}=(\tilde{\alpha}_1x_1,\tilde{\alpha}_2x_2,\dots,\tilde{\alpha}_Nx_N)$. *In particular*

$$\int\frac{\tau\left(M,\mathbf{t},\mathbf{T},\mathbf{t}^{*+}(\alpha\mathbf{y}_{(N)})\right)}{\tau\left(M,\mathbf{0},\mathbf{T},\mathbf{0}\right)}\Delta_{\mathbf{T}}(\alpha\mathbf{y}_{(N)})\prod_{i=1}^{N}d\mu(\alpha_i)=$$
$$\frac{\tau\left(M,\mathbf{t},\mathbf{T}+\mathbf{T}(\mu),\mathbf{t}^{*+}(\mathbf{y}_{(N)})\right)}{\tau\left(M,\mathbf{0},\mathbf{T}+\mathbf{T}(\mu),\mathbf{0}\right)}\Delta_{\mathbf{T}+\mathbf{T}(\mu)}(\mathbf{y}_{(N)}). \quad (3.9.4)$$

Remember that arbitrary linear combination of tau-functions is not a tau-function. Formulae (3.9.3) and also (3.9.4) give the integral representations for the tau-function (3.1.6). It may help to express a tau-function with the help of a more simple one. If we choose the integration measures:

$$\frac{i}{2\pi}\int_C\psi(\mathbf{T},\alpha z)e^{-\alpha}(-\alpha)^{-b-1}d\alpha=\psi(\mathbf{T}+\mathbf{T}^{b},z), \quad (3.9.5)$$

$$\int_0^\infty\psi(\mathbf{T},\alpha z)e^{-\alpha}\alpha^{a}d\alpha=\psi(\mathbf{T}+\mathbf{T}^{a},z), \quad (3.9.6)$$

$$\int_0^1\psi(\mathbf{T},\alpha z)\alpha^{a}(1-\alpha)^{b-a-1}d\alpha=\psi(\mathbf{T}+\mathbf{T}^{c},z), \quad (3.9.7)$$

where $C$ starts at $+\infty$ on the real axis, circles the origin in the counterclockwise direction and returns to the starting point. Then

$$T_n^b=\ln\Gamma(b+n+1),\quad T_n^a=-\ln\Gamma(a+n+1),\quad T_n^c=\ln\frac{\Gamma(b+n+1)}{\Gamma(a+n+1)\Gamma(b-a)}. \quad (3.9.8)$$

Also consider the $q$-integrals [13]:

$$q^{-(a+n)(a+n+1)}\int_0^\infty\psi(\mathbf{T},\alpha(1-q)z)E_q(-\alpha)\alpha^{a}d_q\alpha=\psi\left(\mathbf{T}+\mathbf{T}(a,q),z\right), \quad (3.9.9)$$

$$\frac{1}{\Gamma_q(b-a)}\int_0^1\psi(\mathbf{T},\alpha z)\alpha^{a}\frac{(\alpha q;q)_\infty}{(\alpha q^{b-a};q)_\infty}d_q\alpha=\psi(\mathbf{T}+\mathbf{T}(a,b,q),z). \quad (3.9.10)$$

Then

$$T_n(a,q)=\ln\frac{1}{(1-q)^n\Gamma_q(a+n+1)},\quad T_n(a,b,q)=\ln\frac{\Gamma_q(b+n+1)}{\Gamma_q(a+n+1)}. \quad (3.9.11)$$

In the same way one can consider Miwa change (2.2.15). In the *Examples* below we shall present hypergeometric functions listed in the *Subsections 1.2 and 1.3* as tau-functions of the type (3.2.14). Then we are able to write down integration formulae, namely (3.9.4), which express ${}_{p+1}\Phi_s$ and ${}_{p+1}\Phi_{s+1}$ in terms of ${}_p\Phi_s$ with the help of (3.9.9), (3.9.10) and (3.9.3). In [12] different integral representation formula was presented, which was based on the $q$-analog of Selberg's integral of Askey and Kadell. By taking the limit $q\to1$ one can consider functions ${}_pF_s$. Using (3.9.8), one can express ${}_{p+1}\mathcal{F}_s$, ${}_{p+1}\mathcal{F}_{s+1}$ and ${}_p\mathcal{F}_{s+1}$ as integrals of ${}_p\mathcal{F}_s$ with the help of (3.9.6), (3.9.7) and (3.9.5) respectively.

### 3.10 Examples

The main point of the paper is the observation that if $r(D)$ is a rational function of $D$ then $\tau_r$ is a hypergeometric series. If $r(D)$ is a rational function of $q^D$ we obtain $q$-deformed hypergeometric series. Now let us consider various $r(D)$.

**Example 1** Let $r = 1$. One can put $\mathbf{T} = 0$. Then one gets

$$\tau_{r=1}(M, \mathbf{t}, \mathbf{t}^*) = \exp\left(\sum_{n=1}^{\infty} n t_n t_n^*\right), \tag{3.10.1}$$

which is vacuum tau-function for the two-dimensional Toda lattice. Formula (3.10.1) is a manifestation of summation formulas for Schur functions [13]. Let us note that this is also an example of function ${}_1\mathcal{F}_0(1.2.6)$.

**Example 2** Let $r(n) = n$, that is $T_n = \ln \frac{1}{n!}, n \geq 0$ and $T_n = \ln(-1)^n(-n-1)!, n < 0$. Also let us put $\mathbf{t}^* = (t_1^*, 0, 0, ...)$. For $M = 0, \pm 1$ we get

$$\tau_r(0, \mathbf{t}, t_1^*) = 1, \quad \tau_r(1, \mathbf{t}, t_1^*) = e^{\xi(\mathbf{t}, t_1^*)}, \quad \tau_r(-1, \mathbf{t}, t_1^*) = e^{-\xi(\mathbf{t}, t_1^*)}. \tag{3.10.2}$$

Here $t_1^*$ plays the role of spectral parameter for the vacuum Baker-Akhiezer function. This fact is in accordance to the meaning of $t_1^*$ as a group time for the Galilean transformation [25]. Similar answers $\tau = e^{\pm\xi(\mathbf{t}, z^*)}$ one obtains if he substitutes $t_n^* = \pm n^{-1}(z^*)^n$ to (3.10.1). Let us note that (3.10.2) are the functions ${}_1F_0(0; t_1, t_2, \ldots)$, ${}_1F_0(1; t_1, t_2, \ldots)$ and ${}_1F_0(-1; t_1, t_2, \ldots)$ (1.3.2) which will be described below in the *Example 3* (3.10.7).
**Example 3** Let all parameters $b_k$ be nonintegers.

$${}_pr_s(D) = \frac{(D + a_1)(D + a_2) \cdots (D + a_p)}{(D + b_1)(D + b_2) \cdots (D + b_s)}. \tag{3.10.3}$$

If all $a_k$ are also nonintegers the relevant $\mathbf{T}$ is:

$${}^pT_n^s = -\ln \frac{\Gamma(n + a_1 + 1)\Gamma(n + a_2 + 1) \cdots \Gamma(n + a_p + 1)}{\Gamma(n + b_1 + 1)\Gamma(n + b_2 + 1) \cdots \Gamma(n + b_s + 1)}. \tag{3.10.4}$$

For the correlator (3.1.7) we have:

$$\frac{{}^p\tau^s(M, \mathbf{t}, \mathbf{T}, \mathbf{t}^*)}{{}^p\tau^s(M, \mathbf{0}, \mathbf{T}, \mathbf{0})} = {}^p\tau_r^s(M, \mathbf{t}, \mathbf{t}^*) = \sum_{\mathbf{n}} s_{\mathbf{n}}(\mathbf{t}) s_{\mathbf{n}}(\mathbf{t}^*) \frac{(a_1 + M)_{\mathbf{n}} \cdots (a_p + M)_{\mathbf{n}}}{(b_1 + M)_{\mathbf{n}} \cdots (b_s + M)_{\mathbf{n}}}. \tag{3.10.5}$$

If in formula (3.10.5) we put

$$t_1^* = 1, \qquad t_i^* = 0, \quad i > 1 \tag{3.10.6}$$

then $s_{\mathbf{n}}(\mathbf{t}^*) = H_{\mathbf{n}}^{-1}$, and we obtain the hypergeometric function related to Schur functions [13] (see [26] for help):

$$\frac{{}^p\tau^s(M, \mathbf{t}, \mathbf{T}, \mathbf{t}^*)}{{}^p\tau^s(M, \mathbf{0}, \mathbf{T}, \mathbf{0})} = {}^p\tau_r^s(M, \mathbf{t}, \mathbf{t}^*) = {}_pF_s\left(\begin{matrix} a_1 + M, \ldots, a_p + M \\ b_1 + M, \ldots, b_s + M \end{matrix}\middle| t_1, t_2, \ldots\right) =$$
$$\sum_{\mathbf{n}} \frac{(a_1 + M)_{\mathbf{n}} \cdots (a_p + M)_{\mathbf{n}}}{(b_1 + M)_{\mathbf{n}} \cdots (b_s + M)_{\mathbf{n}}} \frac{s_{\mathbf{n}}(\mathbf{t})}{H_{\mathbf{n}}}. \tag{3.10.7}$$

In the last formula $H_{\mathbf{n}}$ is the following hook polynomial (compare with (1.2.5)):

$$H_{\mathbf{n}} = \prod_{(i,j)\in\mathbf{n}} h_{ij}, \qquad h_{ij} = (n_i + n'_j - i - j + 1). \tag{3.10.8}$$

We obtain ordinary hypergeometric function of one variable of type

$${}_{p-1}F_s(a_2, \dots, a_p; b_1, \dots, b_s; \pm t_1 t_1^*) = \tau_r(\pm 1, \mathbf{t}, \mathbf{T}, \mathbf{t}^*), \tag{3.10.9}$$

if we take $a_1 = 0$, $\mathbf{t} = (t_1, 0, 0, \dots)$, $\mathbf{t}^* = (t_1^*, 0, 0, \dots)$ [43].

For variables $\mathbf{t}^+(\mathbf{x}_{(N)})$ the formula (3.10.7) turns out to be

$$\begin{gathered}{}^p\tau_r^s(M, \mathbf{t}^+(\mathbf{x}_{(N)}), \mathbf{t}^*) = {}_pF_s\left( \begin{matrix} a_1 + M, \dots, a_p + M \\ b_1 + M, \dots, b_s + M \end{matrix} \middle| \mathbf{x}_{(N)} \right) = \\ \sum_{\substack{\mathbf{n} \\ l(\mathbf{n})\le N}} \frac{(a_1 + M)_{\mathbf{n}} \cdots (a_p + M)_{\mathbf{n}}}{(b_1 + M)_{\mathbf{n}} \cdots (b_s + M)_{\mathbf{n}}} \frac{s_{\mathbf{n}}(\mathbf{x}_{(N)})}{H_{\mathbf{n}}}.\end{gathered} \tag{3.10.10}$$

We got the hypergeometric function (1.3.2) related to zonal polynomials for the symmetric space $GL(N, C)/U(N)$ [14]. Here $x_i = z_i^{-1}, i = 1, ..., N$ are the eigenvalues of the matrix $\mathbf{X}$, and for zonal spherical polynomials there is the following matrix integral representation

$$Z_{\mathbf{n}}(\mathbf{X}) = Z_{\mathbf{n}}(\mathbf{I}_N) \int_{U(N,C)} \Delta^{\mathbf{n}} (U^* \mathbf{X} U) \, d_* U, \tag{3.10.11}$$

where $\Delta^{\mathbf{n}}(\mathbf{X}) = \Delta_1^{n_1-n_2}\Delta_2^{n_2-n_3}\cdots\Delta_N^{n_N}$ and $\Delta_1, \dots \Delta_N$ are main minors of the matrix $\mathbf{X}$, $d_*U$ is the invariant measure on $U(N, C)$, see [14] for the details.
Taking $N = 1$ we obtain the ordinary hypergeometric function of one variable, which is $x = x_1$ now (compare with (3.10.9)). The ordinary hypergeometric function satisfies well-known hypergeometric equation

$$(\partial_x - {}_pr_s(D))\, {}_pF_s(a_1, \dots, a_p; b_1, \dots, b_s; x) = 0, \quad D := x\partial_x. \tag{3.10.12}$$

This relation helps us to understand the meaning of function $r$.
It is known that the series (3.10.10) diverges if $p > s + 1$ (untill any of $a_i + M$ is nonpositive integer). In case $p = s + 1$ it converges in certain domain in the vicinity of $\mathbf{x}_{(N)} = \mathbf{0}$. For $p < s + 1$ the series (3.10.10) converges for all $\mathbf{x}_{(N)}$. These known facts (see [13]) can be also obtained with the help of the determinant representation (3.8.1) and properties of (1.1.1).

**Example 4**. Hypergeometric function of two sets of variables $\mathbf{x}_{(N)}, \mathbf{y}_{(N)}$ we put

$${}_pr_s(D) = \frac{\prod_{i=1}^p (a_i + D)}{\prod_{i=1}^s (b_i + D)} \frac{1}{N - M + D}, \tag{3.10.13}$$

$$e^{-T_n} = \frac{1}{\Gamma(N - M + n + 1)} \frac{\prod_{i=1}^p \Gamma(a_i + n + 1)}{\prod_{i=1}^s \Gamma(b_i + n + 1)}, \tag{3.10.14}$$

For the variables $\mathbf{t}^+(\mathbf{x}_{(N)})$ and $\mathbf{t}^{*+}(\mathbf{y}_{(N)})$ we obtain (see Section 3 of [26] for help) the formula (1.3.2)

$$\begin{gathered}\langle M| e^{H(\mathbf{t}^+(\mathbf{x}_{(N)}))} e^{A(\mathbf{t}^{*+}(\mathbf{y}_{(N)}))} |M\rangle = {}_p\mathcal{F}_s\left( \begin{matrix} a_1 + M, \dots, a_p + M \\ b_1 + M, \dots, b_s + M \end{matrix} \middle| q, \mathbf{x}_{(N)}, \mathbf{y}_{(N)} \right) = \\ \sum_{\substack{\mathbf{n} \\ l(\mathbf{n})\le N}} \frac{s_{\mathbf{n}}(\mathbf{x}_{(N)}) s_{\mathbf{n}}(\mathbf{y}_{(N)})}{(N)_{\mathbf{n}}} \frac{(a_1 + M)_{\mathbf{n}} \cdots (a_p + M)_{\mathbf{n}}}{(b_1 + M)_{\mathbf{n}} \cdots (b_s + M)_{\mathbf{n}}}.\end{gathered} \tag{3.10.15}$$

**Example 5** The $q$-generalization of the *Example 3*:

$$ {}_pr_s^{(q)}(D)=\frac{\prod_{i=1}^p(1-q^{a_i+D})}{\prod_{i=1}^s(1-q^{b_i+D})}. \tag{3.10.16}$$

For the variables $\mathbf{t}^+(\mathbf{x}_{(N)})$ and

$$ y_k=q^{k-1},\quad k=1,2,...,\quad t_m^*=\sum_{k=1}^{+\infty}\frac{y_k^m}{m}=\frac{1}{m(1-q^m)},\quad m=1,2,\dots \tag{3.10.17}$$

we get Milne's hypergeometric function (1.2.1):

$$ \langle M|e^{H(\mathbf{t})}e^{A(\mathbf{t}^*)}|M\rangle={}_p\Phi_s\left(\begin{matrix}a_1+M,\dots,a_p+M\\ b_1+M,\dots,b_s+M\end{matrix}\middle| q,\mathbf{x}_{(N)}\right)=$$

$$ \sum_{\substack{\mathbf{n}\\ l(\mathbf{n})\le N}}\frac{(q^{a_1+M};q)_{\mathbf{n}}\cdots(q^{a_p+M};q)_{\mathbf{n}}}{(q^{b_1+M};q)_{\mathbf{n}}\cdots(q^{b_s+M};q)_{\mathbf{n}}}\frac{q^{n(\mathbf{n})}}{H_{\mathbf{n}}(q)}s_{\mathbf{n}}(\mathbf{x}_{(N)}). \tag{3.10.18}$$

**Example 6**. To obtain Milne's hypergeometric function of two sets of variables $\mathbf{x}_{(N)},\mathbf{y}_{(N)}$ we use $\mathbf{t}^+(\mathbf{x}_{(N)})$ and $\mathbf{t}^{*+}(\mathbf{y}_{(N)})$. This choice restricts the sum over partitions $\mathbf{n}$ with $l\mathbf{n}\le N$. We put

$$ {}_pr_s^{(q)}(n)=\frac{\prod_{i=1}^p(1-q^{a_i+n})}{\prod_{i=1}^s(1-q^{b_i+n})}\frac{1}{1-q^{N-M+n}}, \tag{3.10.19}$$

$$ e^{-T_n}=\frac{1}{(1-q)^n\Gamma_q(n+N-M+1)}\frac{\prod_{i=1}^p(1-q)^n\Gamma_q(a_i+n+1)}{\prod_{i=1}^s(1-q)^n\Gamma_q(b_i+n+1)}, \tag{3.10.20}$$

$$ \Gamma_q(a)=(1-q)^{1-a}\frac{(q;q)_\infty}{(q^a,q)_\infty},\qquad (q^a,q)_n=(1-q)^n\frac{\Gamma_q(a+n)}{\Gamma_q(a)}. \tag{3.10.21}$$

Here $\Gamma_q(a)$ is a $q$-deformed Gamma-function

$$ \Gamma_q(a)=(1-q)^{1-a}\frac{(q;q)_\infty}{(q^a,q)_\infty},\qquad (q^a,q)_n=(1-q)^n\frac{\Gamma_q(a+n)}{\Gamma_q(a)}. \tag{3.10.22}$$

We obtain (see Section 3 of [26] for help) the Milne's formula (1.3.1)

$$ \tau_r(M,\mathbf{t}^+(\mathbf{x}_{(N)}),\mathbf{t}^{*+}(\mathbf{y}_{(N)}))={}_p\Phi_s\left(\begin{matrix}a_1+M,\dots,a_p+M\\ b_1+M,\dots,b_s+M\end{matrix}\middle| q,\mathbf{x}_{(N)},\mathbf{y}_{(N)}\right)=$$

$$ \sum_{\substack{\mathbf{n}\\ l(\mathbf{n})\le N}}\frac{q^{n(\mathbf{n})}}{H_{\mathbf{n}}(q)}\frac{s_{\mathbf{n}}(\mathbf{x}_{(N)})s_{\mathbf{n}}(\mathbf{y}_{(N)})}{s_{\mathbf{n}}(1,q,\dots,q^{N-1})}\frac{(q^{a_1+M};q)_{\mathbf{n}}\cdots(q^{a_p+M};q)_{\mathbf{n}}}{(q^{b_1+M};q)_{\mathbf{n}}\cdots(q^{b_s+M};q)_{\mathbf{n}}}. \tag{3.10.23}$$

This is the KP tau-function (but not the TL one because (3.10.19) depends on TL variable $M$).

To receive the basic hypergeometric function of one set of variables we must put indeterminates $\mathbf{y}_{(N)}$ in (3.10.18) as $y_i=q^{i-1},i=(1,\dots,N)$. Thus we have

$$ {}_p\Phi_s\left(\begin{matrix}a_1+M,\dots,a_p+M\\ b_1+M,\dots,b_s+M\end{matrix}\middle| q,\mathbf{x}_{(N)}\right)=$$

$$ \sum_{\substack{\mathbf{n}\\ l(\mathbf{n})\le N}}\frac{(q^{a_1+M};q)_{\mathbf{n}}\cdots(q^{a_p+M};q)_{\mathbf{n}}}{(q^{b_1+M};q)_{\mathbf{n}}\cdots(q^{b_s+M};q)_{\mathbf{n}}}\frac{q^{n(\mathbf{n})}}{H_{\mathbf{n}}(q)}s_{\mathbf{n}}(\mathbf{x}_{(N)}). \tag{3.10.24}$$

And for $N=1$ we have the ordinary $q$-deformed hypergeometrical function:

$$
{}_p\Phi_s\left(\begin{matrix} a_1+M,\dots,a_p+M\\ b_1+M,\dots,b_s+M\end{matrix}\middle|\, q,x\right)=\sum_{n=0}^{+\infty}\frac{(q^{a_1+M};q)_n\cdots(q^{a_p+M};q)_n}{(q^{b_1+M};q)_n\cdots(q^{b_s+M};q)_n}\frac{x^n}{(q;q)_n},\quad x=x_1 \tag{3.10.25}
$$

which satisfies the $q$-difference equation (compare it with (3.10.12)

$$
\left(\frac{1}{x}\left(1-q^D\right)-{}_pr_s^{(q)}(D)\right){}_p\Phi_s(a_1,\dots,a_p;b_1,\dots,b_s;q,x)=0,\quad D:=x\partial_x, \tag{3.10.26}
$$

where ${}_pr_s^{(q)}(D)$ is defined by (3.10.16).

For the bosonic representation of hypergeometric function (3.10.25) see [28].
There are various applications for series (3.10.25), for instance see [52],[53] and [54]. Bosonic representation of (3.10.25) was found in [28]. Let us note that operator $q^D$ which acts on fermions $\psi(z)$ was used in [45] in different context.

**Example 7**

Notations and notions for this *Example* we borrowed from [35]. Let $r$ be a rational function of Jackoby theta-functions $\theta(2x\eta|\tau')$, where $\tau'$ is an elliptic modulus:

$$
{}_pr_s^{(\eta)}(n)=\frac{\prod_{i=1}^p\theta(2\eta(a_i+n)|\tau')}{\theta(2\eta(N-M+n)|\tau')\prod_{i=1}^s\theta(2\eta(b_i+n)|\tau')},\, e^{-T_{n-1}}=\frac{\prod_{i=1}^p[a_i]_n}{[N-M]_n\prod_{i=1}^s[b_i]_n} \tag{3.10.27}
$$

Here elliptic Pochhammer's symbol $[a]_n$ is defined in terms of the elliptic number [a]

$$
[a]=\theta(2a\eta|\tau'),\quad [a]_k=[a][a+1][a+2]\cdots[a+k-1]. \tag{3.10.28}
$$

One can associate the elliptic Pochhammer's symbol with a given partition $\mathbf{n}$:

$$
[a]_{\mathbf{n}}=[a]_{n_1}[a-1]_{n_2}\cdots[a-l+1]_{n_l}. \tag{3.10.29}
$$

For the variables $\mathbf{t}^+(\mathbf{x}_{(N)})$ and $\mathbf{t}^{*+}(\mathbf{y}_{(N)})$ we can introduce the hypergeometric function

$$
\langle M|e^{H(\mathbf{t}^+(\mathbf{x}_{(N)}))}e^{A(\mathbf{t}^{*+}(\mathbf{y}_{(N)}))}|M\rangle={}_pF_s^{(\eta)}\left(\begin{matrix} a_1+M,\dots,a_p+M\\ b_1+M,\dots,b_s+M\end{matrix}\middle|\,\eta,\mathbf{x}_{(N)},\mathbf{y}_{(N)}\right)=
$$

$$
\sum_{\substack{\mathbf{n}\\ l(\mathbf{n})\le N}}\frac{s_{\mathbf{n}}(\mathbf{x}_{(N)})s_{\mathbf{n}}(\mathbf{y}_{(N)})}{[N]_{\mathbf{n}}}\frac{[a_1+M]_{\mathbf{n}}\cdots[a_p+M]_{\mathbf{n}}}{[b_1+M]_{\mathbf{n}}\cdots[b_s+M]_{\mathbf{n}}}. \tag{3.10.30}
$$

As in the case of (1.2.1) this the KP tau-function which is not the TL tau-function because the factor $[N]_{\mathbf{n}}$ in the denominator does not depend on $M$. For $N=1$ we get elliptic hypergeometric function of one variable [35]. For instance to obtain the elliptic very-well-poised hypergeometric function

$$
{}_{p+1}W_p(\alpha_1;\alpha_4,\alpha_5,\dots,\alpha_{p+1};z|\eta,\tau')=\sum_{n=0}^{\infty}z^n\frac{[\alpha_1+2n][\alpha_1]_n}{[\alpha_1][n]!}\prod_{m=1}^{p-2}\frac{[\alpha_{m+3}]_n}{[\alpha_1-\alpha_{m+3}+1]_n}, \tag{3.10.31}
$$

we choose

$$
t_n=\frac{z^n}{n},\quad t_n^*=\frac{1}{n},\quad e^{-T_{n-1}}=\frac{[\alpha_1+2n][\alpha_1]_n}{[\alpha_1][n]!}\prod_{m=1}^{p-2}\frac{[\alpha_{m+3}]_n}{[\alpha_1-\alpha_{m+3}+1]_n}. \tag{3.10.32}
$$

**Example 8** The hypergeometric function (1.3.1),(3.10.23) ${}_1\Phi_1\left(\begin{smallmatrix} a\\ b\end{smallmatrix}\middle|\, q,\mathbf{x}_{(N)},\mathbf{y}_{(N)}\right)$ can be degenerated to ${}_1\Phi_0\left(a|\, q,\mathbf{x}_{(N)},\mathbf{y}_{(N)}\right)$ by taking $b\to+\infty$ (remember that $|q|<1$). The limit

$b \to -\infty$ (with the rescaling of times $x_i, y_i \to q^{\frac{b}{2}} x_i, q^{\frac{b}{2}} x_i$) is also of interest. Consider this limit and put $a = N - M$.

Now we get an example of KP tau function (3.1.7) which is not a hypergeometric function. Take $T_n = -\frac{\gamma}{2}(n + \frac{1}{2})^2$, or the same

$$r(D) = q^{-D}, \quad q = e^{-\gamma}, \tag{3.10.33}$$

and rescale the times once more: $t_k = \alpha^k p_k, t_k^* = \alpha^k p_k^*$. We get the series

$$\langle M| e^{H(\mathbf{t})} e^{A(\mathbf{t}^*)} |M\rangle = \sum_{\mathbf{n}} \alpha^{|\mathbf{n}|} e^{\gamma f_2(\mathbf{n})} s_{\mathbf{n}}(\mathbf{p}) s_{\mathbf{n}}(\mathbf{p}^*), \tag{3.10.34}$$

$$f_2(\mathbf{n}) = \frac{1}{2} \sum_i \left[ (n_i - i + M - \frac{1}{2})^2 - (-i + M - \frac{1}{2})^2 \right], \tag{3.10.35}$$

which was recently considered in [60] (our notations $\mathbf{n}, \alpha, \gamma$ are related to $\lambda, q, \beta$ in [60] respectively). This series is a generating function for double Hurwitz numbers $Hur_{d,b}(n, m)$ introduced in [60] as follows. $Hur_{d,b}(n, m)$ is a weighted number of connected degree $d$ covering of $P^1$ with monodromy around $0, \infty \in P^1$ being $n$ and $m$, respectively, and $b$ additional simple ramifications. The genus of each covering is $g = (b + 2 - l(n) - l(m))/2$, where $l(n)$ is the number of parts of $n$. The weight of each covering is the reciprocal of the order of its automorphism group. The formula presented in [60] in our terms reads as

$$\log \langle M| e^{H(\mathbf{t})} e^{A(\mathbf{t}^*)} |M\rangle = \sum_{d,b,n,m} \alpha^d \gamma^b p_n p_m^* Hur_{d,b}(n, m)/b!, \tag{3.10.36}$$

for $A$ see (3.1.3),(3.1.1),(3.10.33). Therefore the generating function for the double Hurwitz numbers is expressed in terms of group cocycle of the $\Psi DO$ on the circle (see *Appendix "Gauss factorization problem, additional symmetries, string equations and* $\Psi DO$ *on the circle*), which is the correlator under the logarithm.

**Example 9**. Take $r$ be a step function: $r(n) = 0, n < k$ and $r(n) = 1, n \geq k$, and let $k < N$. Then

$$\tau_r(\mathbf{x}_{(N)}, \mathbf{y}_{(N)}) = \sum_{\substack{\mathbf{n} \\ l(\mathbf{n}) \leq k < N}} s_{\mathbf{n}}(\mathbf{x}_{(N)}) s_{\mathbf{n}}(\mathbf{y}_{(N)}) \tag{3.10.37}$$

is equal the determinant of a Toeplitz matrix - it is a subject of Gessel's theorem. We have the determinant representation of (3.10.37) due to the formula (3.8.1).

## 3.11 Baker-Akhiezer functions and Sato Grassmannian

Let us write down the expression for Baker-Akhiezer functions (2.3.5) in terms of Miwa variables (2.2.15):

$$w_\infty(M, \mathbf{t}^-(\mathbf{x}_{(N)}), \mathbf{t}^*, \frac{1}{z}) = \frac{\tau_r(M, \mathbf{t}^-(\mathbf{x}_{(N+1)}), \mathbf{t}^*)}{\tau_r(M, \mathbf{t}^-(\mathbf{x}_{(N)}), \mathbf{t}^*)} \prod_{i=1}^{N} (1 - \frac{x_i}{z}), \quad \mathbf{x}_{(N+1)} = (x_1, \dots, x_N, z), \tag{3.11.1}$$

$$w_\infty^*(M, \mathbf{t}, \mathbf{t}^*, \frac{1}{z}) = -\frac{\tau_r(M, \mathbf{t} + [\mathbf{z}], \mathbf{t}^*)}{\tau_r(M, \mathbf{x}_{(N)}, \mathbf{t}^*)} \prod_{i=1}^{N} \frac{1}{(1 - \frac{x_i}{z})} \frac{dz}{z}, \quad [\mathbf{z}] = (z, \frac{z^2}{2}, \dots). \tag{3.11.2}$$

We see that the variables $x_k, k = 1, \dots, N$ are zeroes of $w_\infty(z)$ and poles of $w_\infty^*(z)$.

**Remark 3** *The associated linear problems for Baker-Akhiezer functions are read as*

$$(\partial_{t_1} - \partial_{t_1}\phi_n)\, w(n, \mathbf{t}, \mathbf{t}^*, z) = w(n+1, \mathbf{t}, \mathbf{t}^*, z), \tag{3.11.3}$$

$$\partial_{t_1^*} w(n, \mathbf{t}, \mathbf{t}^*, z) = r(n) e^{\phi_{n-1}-\phi_n} w(n-1, \mathbf{t}, \mathbf{t}^*, z). \tag{3.11.4}$$

*where $w$ is either $w_\infty$ or $w_0$. The compatibility of these equations gives rise to the equation (3.3.8). Taking into account the second eq.(3.3.8), equations (3.11.3), (3.11.4) may be also viewed as the recurrent equations for the tau-functions which depend on different number of variables $\mathbf{x}_{(N)}$.*

Let us write down a plane of Baker-Akhiezer functions (2.3.5), which characterizes Sato Grassmannian related to the tau-function (3.3.1) $\tau_r(M, \mathbf{t}, \mathbf{t}^*)$. We take $x_k = 0, k = 1, \ldots, N$ in (3.11.1) and obtain:

$$w_\infty(n, \mathbf{0}, \mathbf{t}^*, z) = z^n (1 + \sum_{m=1}^{\infty} r(n) r(n-1) \cdots r(n-m+1) p_m(-\mathbf{t}^*) z^{-m}), \quad n = M, M+1, M+2, .... \tag{3.11.5}$$

The dual plane is

$$w_\infty^*(n, \mathbf{0}, \mathbf{t}^*, z) = z^{-n} (1 + \sum_{m=1}^{\infty} r(n) r(n+1) \cdots r(n+m-1) p_m(\mathbf{t}^*) z^{-m}) dz, \quad n = M, M+1, M+2, .... \tag{3.11.6}$$

About these formulae see also (A4.22),(A4.23).
We see that when $r$ has zeroes, then in the regions (3.5.3) the Grassmannian is the finite-dimensional one. The corresponding tau-function is a particular case of the one found in [38], [39].

## 3.12 Different representations

Let us rewrite hypergeometric series in different way representing all Pochhammer's coefficients $(q^a; q)_{\mathbf{n}}$ and $(a)_{\mathbf{n}}$ through Schur functions. This gives us the opportunity to interchange the role of Pochhammer's coefficients and Schur functions in (1.3.1),(3.10.7), and to present different fermionic representations of the hypergeometric functions. We have the relations (see [26]):

$$\prod_{(i,j)\in\mathbf{n}} (1 - q^{a+j-i}) = \frac{s_{\mathbf{n}}(\mathbf{t}(a,q))}{s_{\mathbf{n}}(\mathbf{t}(+\infty, q))}, \quad \prod_{(i,j)\in\mathbf{n}} (a + j - i) = \frac{s_{\mathbf{n}}(\mathbf{t}(a))}{s_{\mathbf{n}}(\mathbf{t}(+\infty)}, \tag{3.12.1}$$

where parameters $t_m(a,q)$ and $t_m(a)$ are chosen via generalized Miwa transform [11] with multiplicity $a$ (remember that $|q| < 1$)

$$t_m(a,q) = \frac{1 - (q^a)^m}{m(1 - q^m)}, \quad t_m(a) = \frac{a}{m}, \quad m = 1, 2, \ldots, \tag{3.12.2}$$

$$s_{\mathbf{n}}(\mathbf{t}(+\infty, q)) = \lim_{a\to+\infty} s_{\mathbf{n}}(\mathbf{t}(a,q)) = \frac{q^{n(\mathbf{n})}}{H_{\mathbf{n}}(q)}, \tag{3.12.3}$$

$$s_{\mathbf{n}}(\mathbf{t}(+\infty)) = \lim_{a\to+\infty} s_{\mathbf{n}}\left(\frac{t_1(a)}{a}, \frac{t_2(a)}{a^2}, \ldots\right) = \lim_{a\to+\infty} \frac{1}{a^{|\mathbf{n}|}} s_{\mathbf{n}}(\mathbf{t}(a)) = \frac{1}{H_{\mathbf{n}}}. \tag{3.12.4}$$

Now we rewrite the series (3.10.18) and (3.10.7) only in terms of Schur functions:

$$ {}_p\Phi_s \left( \begin{matrix} a_1 + M, \ldots, a_p + M \\ b_1 + M, \ldots, b_s + M \end{matrix} \middle| q, \mathbf{x}_{(N)}, \mathbf{y}_{(N)} \right) = \tau_r(M, \mathbf{t}(+\infty, q), \mathbf{t}^*)$$

$$= \sum_{\substack{\mathbf{n} \\ l(\mathbf{n})\le N}} \frac{\prod_{k=1}^p s_{\mathbf{n}}(\mathbf{t}(a_k + M, q))}{\prod_{k=1}^s s_{\mathbf{n}}(\mathbf{t}(b_k + M, q))} \left(s_{\mathbf{n}}(\mathbf{t}(+\infty, q))\right)^{s-p+1} \frac{s_{\mathbf{n}}(\mathbf{x}_{(N)}) s_{\mathbf{n}}(\mathbf{y}_{(N)})}{s_{\mathbf{n}}(\mathbf{t}(N,q))}, \tag{3.12.5}$$

$$
{}_p\mathcal{F}_s\left(\begin{matrix} a_1+M,\ldots,a_p+M \\ b_1+M,\cdots,b_s+M \end{matrix}\middle| \mathbf{x}_{(N)},\mathbf{y}_{(N)}\right) = \tau_r(M,\mathbf{t}(+\infty),\mathbf{t}) =
$$

$$
\sum_{\substack{\mathbf{n} \\ l(\mathbf{n})\le N}} \frac{\prod_{k=1}^{p} s_{\mathbf{n}}(\mathbf{t}(a_k+M))}{\prod_{k=1}^{s} s_{\mathbf{n}}(\mathbf{t}(b_k+M))} \left(s_{\mathbf{n}}(\mathbf{t}(+\infty))\right)^{s-p+1} \frac{s_{\mathbf{n}}(\mathbf{x}_{(N)}) s_{\mathbf{n}}(\mathbf{y}_{(N)})}{s_{\mathbf{n}}(\mathbf{t}(N))}. \tag{3.12.6}
$$

A nice feature of this formulae is that they do not contain number coefficients at all, it is a sum of ratios of Schur functions only.

We obtain different fermionic representations of hypergeometric functions (3.12.6), (3.12.5), and they are parametrized by a complex noninteger number $b$:

**Proposition 10** *For $b \in C$ and for $r = {}_pr_s$ (see (3.10.3)) we have*

$$
\tau_r(M,\mathbf{t}(+\infty),\mathbf{t}^*) = \tau_{r_b}(M,\mathbf{t}(b+M),\mathbf{t}^*), \quad r_b = \frac{r}{b+D}. \tag{3.12.7}
$$

*For $r = {}_pr_s^{(q)}$ (see (3.10.16)) we have*

$$
\tau_r(M,\mathbf{t}(+\infty,q),\mathbf{t}^*) = \tau_{r_b}(M,\mathbf{t}(b+M,q),\mathbf{t}^*), \quad r_b = \frac{r}{1-q^{b+D}}. \tag{3.12.8}
$$

**Remark 4** *There are two ways to restrict the sum (3.1.7) to the sum over partitions of length $l(\mathbf{n}) \le N$. First, if we use Miwa's change (2.2.14), then $s_{\mathbf{n}}(\mathbf{x}_{(N)}) = 0$, for $\mathbf{n}$ with length $l(\mathbf{n}) > N$. The second way is to restrict the Pochhammer's coefficients: if we put $a_i = N$ for one $i$ from (3.10.16) equal to $N$, then the coefficient $(q^{a_i},q)_{\mathbf{n}}$ vanishes for $l(\mathbf{n}) > N$. Since we expressed Pochhammer's coefficients in terms of Schur functions in (3.12.1) both ways have the same explanation. Indeed*

$$
t_m(N,q) = \frac{1}{m}\frac{1-(q^N)^m}{1-q^m} = \frac{1}{m}(1+(q)^m+(q^2)^m+\cdots+(q^{N-1})^m). \tag{3.12.9}
$$

*Therefore we obtain for Miwa's change: $x_1 = 1, x_2 = q, \ldots, x_N = q^{N-1}$ and*

$$
s_{\mathbf{n}}(\mathbf{t}(N,q)) = s_{\mathbf{n}}(1,q,\ldots,q^{N-1}) = 0, \quad l(\mathbf{n}) > N. \tag{3.12.10}
$$

*The same we have for the sum over partitions $\mathbf{n}$ such that $l(\mathbf{n}') < K$. Again the first way has to be realized through the following Miwa's change of variables:*

$$
t_m = -\sum_{i=1}^{K} \frac{x_i^m}{m}, \quad s_{\mathbf{n}}(\mathbf{t}) = s_{\mathbf{n}'}(\mathbf{x}_{(K)}). \tag{3.12.11}
$$

*The second way is to make one of the parameters, for example $a_j$ from (3.10.16) equal to $(-K)$. In this case*

$$
s_{\mathbf{n}}(\mathbf{t}(-K,q)) = s_{\mathbf{n}'}\left(\frac{1}{q},\frac{1}{q^2},\ldots,\frac{1}{q^K}\right) = 0, \quad l(\mathbf{n}') > K. \tag{3.12.12}
$$

# 4 Further generalization. Examples of Gelfand-Graev hypergeometric functions

## 4.1 Generalization

Formula (3.3.7) is related to 'Gauss decomposition' of operators inside vacuums $\langle 0|\dots|0\rangle$ into diagonal operator $e^{H_0(\mathbf{T})}$ and upper triangular operator $e^{H(\mathbf{t})}$ and lower triangular operator $e^{-H^*(\mathbf{t}^*)}$ the last two have the Toeplitz form. Now let us consider more general two-dimensional Toda chain tau-function

$$\tau = \langle M|e^{H(\mathbf{t})}ge^{-A(\mathbf{t}^*)}|M\rangle, \tag{4.1.1}$$

where we decompose $g$ in the following way:

$$g(\tilde{\gamma},\gamma) = e^{\tilde{A}_1(\tilde{\gamma}_1)}\cdots e^{\tilde{A}_k(\tilde{\gamma}_k)}e^{-A_l(\gamma_l)}\cdots e^{-A_1(\gamma_1)}, \tag{4.1.2}$$

where each of $\tilde{\gamma}_i, \gamma_i, \tilde{A}_i, A_i$ has an additional index: $\tilde{\gamma}_{in}, \gamma_{in}, \tilde{A}_{in}, A_{in}$, $(n = 1, 2, \dots)$. Here each of $A_i(\gamma_i)$ has a form as in (3.1.1),(3.1.3) and corresponds to operator $r^i(D)$, while each of $\tilde{A}_i(\tilde{\gamma}_i)$ has a form of (3.1.11) and corresponds to operator $\tilde{r}^i(D)$. Collections of variables $\tilde{\gamma} = \{\tilde{\gamma}_{in}\}, \gamma = \{\gamma_{in}\}$ play the role of coordinates for some wide enough class of Clifford group elements $g$. This tau-function is related to rather involved generalization of the hypergeometric functions we considered above. Tau-function (4.1.1),(4.1.2) may be considered as the result of applying of the additional symmetries to the vacuum tau function, which is 1, see *Appendix "The vertex operator action"*.

Let us calculate this tau-function. First of all we introduce a set consisting of $m+1$ partitions:

$$(\mathbf{n_1}, \dots, \mathbf{n_m}, \mathbf{n_{m+1}} = \mathbf{n}), \quad 0 \le \mathbf{n_1} \le \mathbf{n_2} \le \dots \le \mathbf{n_m} \le \mathbf{n_{m+1}} = \mathbf{n}, \tag{4.1.3}$$

see [26] for the notation $\le$ for the partitions. The corresponding set

$$\Theta^m_{\mathbf{n}} = (\mathbf{n_1}, \theta_1, \dots, \theta_m), \quad \theta_i = \mathbf{n_{i+1}} - \mathbf{n_i}, \quad i = 1, \dots, m \tag{4.1.4}$$

depends on the partition $\mathbf{n}$ and the number $m+1$ of the partitions. We take as $s_\Theta(\mathbf{t}^*, \gamma)$ the product which is relevant to the set $\Theta^m_{\mathbf{n}}$ and depending on the set of variables $\mu_i = \{\mu_{ij}\}$ $(i = (1, \dots, m+1)$, $j = (1, 2 \dots))$

$$s_{\Theta^m_{\mathbf{n}}}(\mu) = s_{\mathbf{n_1}}(\mu_1)s_{\theta_1}(\mu_2)\cdots s_{\theta_m}(\mu_{m+1}). \tag{4.1.5}$$

Here $s_{\theta_i}$ is a skew Schur function (see [26]). Further we define function $r_{\Theta^m_{\mathbf{n}}}(M)$:

$$r_{\Theta^m_{\mathbf{n}}}(M) = r_{\mathbf{n_1}}(M)r^1_{\theta_1}(M)\cdots r^m_{\theta_m}(M), \tag{4.1.6}$$

where the function $r^i_{\theta_i}(M)$ , a skew analogy of $r_{\mathbf{n}}(M)$ from (3.1.4), is

$$r_{\theta_i}(M) = \prod_{j=1}^{s} r(n_j^{(i)} - j + 1 + M)\cdots r(n_j^{(i+1)} - j + M), \tag{4.1.7}$$

where $\mathbf{n_{i+1}} = (n_1^{(i+1)}, \dots, n_s^{(i+1)})$. If the function $r^i(m)$ has no poles and zeroes at integer points then the relation

$$r^i_{\theta_i}(M) = \frac{r^i_{\mathbf{n_{i+1}}}(M)}{r^i_{\mathbf{n_i}}(M)}, \quad i = 1, \dots, m \tag{4.1.8}$$

is correct. To calculate the tau function we need the *Lemma*

**Lemma 3** Let partitions $\mathbf{n} = (i_1, \dots, i_s | j_1 - 1, \dots, j_s - 1)$ and $\tilde{\mathbf{n}} = (\tilde{i}_1, \dots, \tilde{i}_r | \tilde{j}_1 - 1, \dots, \tilde{j}_r - 1)$ satisfy the relation $\mathbf{n} \ge \tilde{\mathbf{n}}$. The following is valid:

$$\langle 0|\psi^*_{\tilde{i}_1}\cdots\psi^*_{\tilde{i}_r}\psi_{-\tilde{j}_r}\cdots\psi_{-\tilde{j}_1}e^{A^i(\gamma_i)}\psi^*_{-j_1}\cdots\psi^*_{-j_s}\psi_{i_s}\cdots\psi_{i_1}|0\rangle =$$
$$= (-1)^{\tilde{j}_1+\dots+\tilde{j}_r+j_1+\dots+j_s}s_\theta(\gamma_i)r_\theta(0), \qquad \theta = \mathbf{n} - \tilde{\mathbf{n}}. \tag{4.1.9}$$

**Proof:** the proof is achieved by direct calculation (see Example 22 in Sec 5 of [26] for help). Then we obtain the generalization of *Proposition 1*:

**Proposition 11**

$$\tau_M(\mathbf{t},\mathbf{t}^*;\gamma,\tilde\gamma)=\sum_{\mathbf{n}}\sum_{\Theta^k_{\mathbf{n}}}\sum_{\Theta^l_{\mathbf{n}}}\tilde r_{\Theta^k_{\mathbf{n}}}(M)r_{\Theta^l_{\mathbf{n}}}(M)s_{\Theta^k_{\mathbf{n}}}(\mathbf{t},\tilde\gamma)s_{\Theta^l_{\mathbf{n}}}(\mathbf{t}^*,\gamma),\tag{4.1.10}$$

*where $\tilde r_{\Theta^k_{\mathbf{n}}}(M)$ and $r_{\Theta^l_{\mathbf{n}}}(M)$ are given by (4.1.7).*

With the help of this series one can obtain different hypergeometric functions.

## 4.2 The example of Gelfand,Graev and Retakh hypergeometric series

Let us consider the tau function:

$$\tau(M,\tilde\beta,\beta;\gamma)=\langle M|e^{\tilde A(\tilde\beta)}e^{-A_l(\gamma_l)}\cdots e^{-A_1(\gamma_1)}e^{-A(\beta)}|M\rangle.\tag{4.2.1}$$

We put

$$\tilde\beta=(x,\frac{x^2}{2},\frac{x^3}{3},\ldots),\quad \beta=(y_1,0,0,\ldots),\quad \gamma_i=(y_{i+1},0,0,\ldots)\quad i=(1,\ldots,l).\tag{4.2.2}$$

We obtain the series

$$\tau(M,x,y_1,\ldots,y_{l+1})=\sum_{n_1,\ldots,n_{l+1}=0}^{+\infty}\tilde r_{(n_1+\cdots+n_{l+1})}(M)r_{\Theta^l_{\mathbf{n}}}(M)\frac{(xy_1)^{n_1}\cdots(xy_{l+1})^{n_{l+1}}}{n_1!\cdots n_{l+1}!}=\tag{4.2.3}$$

$$\sum_{n_1,\ldots,n_{l+1}\in Z}c(n_1,\ldots,n_{l+1})(xy_1)^{n_1}\cdots(xy_{l+1})^{n_{l+1}},c(n_1,\ldots,n_{l+1})=\frac{\tilde r_{(n_1+\cdots+n_{l+1})}(M)r_{\Theta^l_{\mathbf{n}}}(M)}{\Gamma(n_1+1)\cdots\Gamma(n_{l+1}+1)},\tag{4.2.4}$$

where $\Theta^l_{\mathbf{n}}$ corresponds to the set of simple partitions-rows

$$\mathbf{n_1}=(n_1),\mathbf{n_2}=(n_1+n_2),\ldots,\mathbf{n}_{l+1}=(n_1+\cdots+n_{l+1})\tag{4.2.5}$$

When functions $b_i(n_1,\ldots,n_{l+1})$ defined as

$$b_i(n_1,\ldots,n_{l+1})=\frac{c(n_1,\ldots,n_i+1,\ldots,n_{l+1})}{c(n_1,\ldots,n_{l+1})},\quad i=1,\ldots,l+1\tag{4.2.6}$$

are rational functions of $(n_1,\ldots,n_{l+1})$, then tau function (4.2.3) is a Horn hypergeometric series [13].
Above series for the special choice of functions $r^i(D)$ can be deduced from the Gelfand, Graev and Retakh series defined on the special lattice and corresponding to the special set of parameters. Let us take the rational functions $r^i(D)$:

$$r^i(D)=\frac{\prod_{j=1}^{p^{(i)}}(D+a^{(i)}_j)}{\prod_{m=1}^{s^{(i)}}(D+b^{(i)}_m)},\quad (i=0,\ldots,l),\qquad r^0(D)=r(D)\tag{4.2.7}$$

$$\tilde r(D)=\frac{\prod_{j=1}^{p^{(l+1)}}(D+a^{(l+1)}_j)}{\prod_{m=1}^{s^{(l+1)}}(D+b^{(l+1)}_m)}\tag{4.2.8}$$

Let define $N=p^{(0)}+s^{(0)}+2\sum_{j=1}^{l}(p^{(j)}+2s^{(j)})+p^{(l+1)}+s^{(l+1)}+l+1$ and consider complex space $C^N$. In this space we consider the $l+1$-dimensional basis $B$ and the vector $\upsilon$ consisting of parameters.

$$p_0=s_0=0,\quad p_i=p^{(i-1)}+s^{(i)},\quad s_i=s^{(i-1)}+p^{(i)},\quad i=(1,\ldots,l)$$

$$p_{l+1}=p^{(l)}+p^{(l+1)},\quad s_{l+1}=s^{(l)}+s^{(l+1)},\quad N=\sum_{j=1}^{l+1}(p_j+s_j)+l+1\tag{4.2.9}$$

$$\mathbf{f}^i = -(\mathbf{e}_{p_0+s_0+\cdots+p_{i-1}+s_{i-1}+1} + \cdots + \mathbf{e}_{p_1+s_1+\cdots+p_{i-1}+s_{i-1}+p_i}) + \\ +(\mathbf{e}_{p_1+s_1+\cdots+p_{i-1}+s_{i-1}+p_i+1} + \cdots + \mathbf{e}_{p_1+s_1+\cdots+p_{i-1}+s_{i-1}+p_i+s_i}), \quad i = 1, \ldots, l+1 \tag{4.2.10}$$

where $\mathbf{e}_i = \underbrace{(0, \ldots, 0, \hat{\overset{i}{1}}, 0, \ldots)}_{\text{N}}$. The lattice $B \in C^N$ is generated by the vector basis of dimension $l+1$:

$$\mathbf{b}^i = \mathbf{f}^i + \cdots + \mathbf{f}^{l+1} + \mathbf{e}_{N-l-1+i}, \qquad i = 1, \ldots, l+1 \tag{4.2.11}$$

Vector $\upsilon \in C^N$ is defined as follows (compare with (4.2.10)):

$$\upsilon^i = -(a_1^{(i-1)}\mathbf{e}_{p_0+s_0+\cdots+p_{i-1}+s_{i-1}+1} + \cdots + a_{p^{(i-1)}}^{(i-1)}\mathbf{e}_{p_0+s_0+\cdots+p_{i-1}+s_{i-1}+p^{(i-1)}} + \\ +b_1^{(i)}\mathbf{e}_{p_0+s_0+\cdots+p_{i-1}+s_{i-1}+p^{(i-1)}+1} + \cdots + b_{s^{(i)}}^{(i)}\mathbf{e}_{p_0+s_0+\cdots+p_{i-1}+s_{i-1}+p_i}) + \\ +((b_1^{(i-1)}-1)\mathbf{e}_{p_0+s_0+\cdots+p_{i-1}+s_{i-1}+p_i+1} + \cdots + (b_{s^{(i-1)}}^{(i-1)}-1)\mathbf{e}_{p_0+s_0+\cdots+p_{i-1}+s_{i-1}+p_i+s^{(i-1)}} + \\ +(a_1^{(i)}-1)\mathbf{e}_{p_0+s_0+\cdots+p_{i-1}+s_{i-1}+p_i+s^{(i-1)}+1} + \cdots + (a_{s^{(i)}}^{(i)}-1)\mathbf{e}_{p_0+s_0+\cdots+p_{i-1}+s_{i-1}+p_i+s_i}) \tag{4.2.12}$$

for $i = (1, \ldots l)$, and

$$\upsilon^{l+1} = -(a_1^{(l)}\mathbf{e}_{p_0+s_0+\cdots+p_l+s_l+1} + \cdots + a_{p^{(l)}}^{(l)}\mathbf{e}_{p_0+s_0+\cdots+p_l+s_l+p^{(l)}} + \\ +a_1^{(l+1)}\mathbf{e}_{p_0+s_0+\cdots+p_l+s_l+p^{(l)}+1} + \cdots + a_{s^{(l+1)}}^{(l+1)}\mathbf{e}_{p_0+s_0+\cdots+p_l+s_l+p_{l+1}}) + \\ +((b_1^{(l)}-1)\mathbf{e}_{p_0+s_0+\cdots+p_l+s_l+p_{l+1}+1} + \cdots + (b_{s^{(l)}}^{(l)}-1)\mathbf{e}_{p_0+s_0+\cdots+p_l+s_l+p_{l+1}+s^{(l)}} + \\ +(b_1^{(l+1)}-1)\mathbf{e}_{p_0+s_0+\cdots+p_l+s_l+p_{l+1}+s^{(l)}+1} + \cdots + (b_{s^{(l+1)}}^{(l+1)}-1)\mathbf{e}_{p_0+s_0+\cdots+p_l+s_l+p_{l+1}+s_{l+1}}) \tag{4.2.13}$$

Vector $\upsilon$ is:

$$\upsilon = \upsilon^1 + \cdots + \upsilon^{l+1} \tag{4.2.14}$$

Now we can write down Gelfand, Graev and Retakh hypergeometric series corresponding to the lattice $B$ and vector $\upsilon$:

$$F_B(\upsilon; z) = \sum_{\mathbf{b}\in B}\prod_{j=1}^{N}\frac{z_j^{\upsilon_j+b_j}}{\Gamma(\upsilon_j+b_j+1)} + \\ \sum_{n_1,\ldots,n_{l+1}\in Z}\prod_{j=1}^{N}\frac{z_j^{\upsilon_j+n_1b_j^1+\cdots+n_{l+1}b_j^{l+1}}}{\Gamma(\upsilon_j+n_1b_j^1+\cdots+n_{l+1}b_j^{l+1}+1)} \tag{4.2.15}$$

Let us compare this series with tau function (4.2.3):

$$F_B(\upsilon; \mathbf{z}) = c_1(a,b)g_1(\mathbf{z})\cdots c_{l+1}(a,b)g_{l+1}(\mathbf{z})\tau(M, x, y_1, \ldots, y_{l+1}) \tag{4.2.16}$$

where

$$c_i^{-1}(a,b) = \Gamma(1-a_1^{(i-1)})\cdots\Gamma(1-a_{p^{(i-1)}}^{(i-1)})\Gamma(1-b_1^{(i)})\cdots\Gamma(1-b_{s^{(i)}}^{(i)}) \times \\ \times\Gamma(b_1^{(i-1)})\cdots\Gamma(b_{s^{(i-1)}}^{(i-1)})\Gamma(a_1^{(i)})\cdots\Gamma(a_{s^{(i)}}^{(i)}), \quad i = 1, \ldots, l \tag{4.2.17}$$

$$c_{l+1}^{-1}(a,b) = \Gamma(1-a_1^{(l)})\cdots\Gamma(1-a_{p^{(l)}}^{(l)})\Gamma(1-a_1^{(l+1)})\cdots\Gamma(1-a_{s^{(l+1)}}^{(l+1)}) \times \\ \times\Gamma(b_1^{(l)})\cdots\Gamma(b_{s^{(l)}}^{(l)})\Gamma(b_1^{(l+1)})\cdots\Gamma(b_{s^{(l+1)}}^{(l+1)}) \tag{4.2.18}$$

$$\frac{z_{p_0+s_0+\cdots+p_{i-1}+s_{i-1}+p_i+1}\cdots z_{p_1+s_1+\cdots+p_{i-1}+s_{i-1}+p_i+s_i}}{(-z_{p_0+s_0+\cdots+p_{i-1}+s_{i-1}+1})\cdots(-z_{p_0+s_0+\cdots+p_{i-1}+s_{i-1}+p_i})}=1, \quad i=2,\dots,l \tag{4.2.19}$$

$$\frac{z_{p_1+1}\cdots z_{p_1+s_1}z_{N-l}}{(-z_1)\cdots(-z_{p_1})}=y_1 \tag{4.2.20}$$

$$y_i=z_{N-l-1+i}, \quad i=2,\dots,l+1 \tag{4.2.21}$$

$$\frac{z_{p_1+s_1+\cdots+p_l+s_l+p_{l+1}+1}\cdots z_{p_1+s_1+\cdots+p_l+s_l+p_{l+1}+s_{l+1}}}{(-z_{p_1+s_1+\cdots+p_l+s_l+1})\cdots(-z_{p_1+s_1+\cdots+p_l+s_l+p_{l+1}})}=x \tag{4.2.22}$$

$$\begin{aligned} g_i(\mathbf{z}) = {} & z_{p_0+s_0+\cdots+p_{i-1}+s_{i-1}+1}^{\left(-a_1^{(i-1)}\right)}\cdots z_{p_0+s_0+\cdots+p_{i-1}+s_{i-1}+p^{(i-1)}}^{\left(-a_{p^{(i-1)}}^{(i-1)}\right)}\times \\ & \times z_{p_0+s_0+\cdots+p_{i-1}+s_{i-1}+p^{(i-1)}+1}^{\left(-b_1^{(i)}\right)}\cdots z_{p_0+s_0+\cdots+p_{i-1}+s_{i-1}+p_i}^{\left(-b_{s^{(i)}}^{(i)}\right)}\times \\ & \times z_{p_0+s_0+\cdots+p_{i-1}+s_{i-1}+p_i+1}^{\left(b_1^{(i-1)}-1\right)}\cdots z_{p_0+s_0+\cdots+p_{i-1}+s_{i-1}+p_i+s^{(i-1)}}^{\left(b_{s^{(i-1)}}^{(i-1)}-1\right)}\times \\ & \times z_{p_0+s_0+\cdots+p_{i-1}+s_{i-1}+p_i+s^{(i-1)}+1}^{\left(a_1^{(i)}-1\right)}\cdots z_{p_0+s_0+\cdots+p_{i-1}+s_{i-1}+p_i+s_i-1}^{\left(a_{s^{(i)}}^{(i)}-1\right)} \end{aligned} \tag{4.2.23}$$

for $i=(1,\dots l)$, and

$$\begin{aligned} g_{l+1}(\mathbf{z}) = {} & z_{p_1+s_1+\cdots+p_l+s_l+1}^{\left(--a_1^{(l)}\right)}\cdots z_{p_1+s_1+\cdots+p_l+s_l+p^{(l)}}^{\left(-a_{p^{(l)}}^{(l)}\right)}\times \\ & \times z_{p_1+s_1+\cdots+p_l+s_l+p^{(l)}+1}^{\left(-a_1^{(l+1)}\right)}\cdots z_{p_1+s_1+\cdots+p_l+s_l+p_{l+1}}^{\left(-a_{s^{(l+1)}}^{(l+1)}\right)}\times \\ & \times z_{p_1+s_1+\cdots+p_l+s_l+p_{l+1}+1}^{\left(b_1^{(l)}-1\right)}\cdots z_{p_1+s_1+\cdots+p_l+s_l+p_{l+1}+s^{(l)}}^{\left(b_{s^{(l)}}^{(l)}-1\right)}\times \\ & \times z_{p_1+s_1+\cdots+p_l+s_l+p_{l+1}+s^{(l)}+1}^{\left(b_1^{(l+1)}-1\right)}\cdots z_{p_1+s_1+\cdots+p_l+s_l+p_{l+1}+s_{l+1}-1}^{\left(b_{s^{(l+1)}}^{(l+1)}-1\right)} \end{aligned} \tag{4.2.24}$$

# Conclusion

We get multivariable hypergeometric functions as certain tau-functions of the KP hierarchy. It means that we have a set of new relations on the multivariable hypergeometric functions. For instance all hypergeometric functions of the form (3.1.6) or of the form (4.1.10) satisfy bilinear Hirota equations [2] of the KP hierarchy. Hypergeometric functions may be also considered as ratios of the tau-functions of the two-dimensional Toda lattice evaluated at special values of Toda lattice times. To get hypergeometric functions of a single set of arguments (1.2.6),(1.2.1) one should take $\mathbf{t}^*$ as in (3.10.6), (3.10.17). To get hypergeometric functions of a double set of arguments (1.3.1), (1.3.2) one should keep the descrete time $M$ constant. One can get the fermionic representations for different special functions and polynomials related to these hypergeometric functions. Using integral representation one can express hypergeometric functions as the integral of rather simple hypergeometric function. We also get determinant representation of (3.1.6), which may allow to analize analytical properties of multivariable tau-functions in terms of functions of only one variable. We wrote down the system of linear equations on tau-function (3.1.6), which may allow to find applications to quantum mechanical

problems. It is quite unexpected that we obtain $q$-deformed version of these hypergeometric functions as tau-functions not of a $q$-deformed KP hierarchy [46],[47],[48],[49],[50] but of the usual KP hierarchy. It is now an interesting problem to establish links between these results and group-theoretic approach to the $q$-special functions [14, 13] and matrix integrals. Let us note a certain similarity of some of our formulas and formulas from [29]. We expect to work out connections with matrix models of Kontsevich type [30] and two-matrix models related to 2D Toda lattice [31, 32]. We consider (in [42, 43, 44]) the multicomponent KP [10, 40] as a basis for obtaining new examples of hypergeometric functions.

Let us note the paper [60] which turns to be different example of the tau function (3.3.1). This tau-function is not of hypergeometric type, but is closely related to it. This tau-function has an interesting meaning in the algebraic geometry. In [61] the interpretation of the Schur functions as a measure on partitions is explained, thus we have an additional interpretation of hypergeometric functions as generating functions for certain probabilities described in [61].

In the present version of the paper we add references to the [62],[63] where different examples of special functions of one and of two variables were considered as solutions of Hirota equations with variable coefficients. It will be interesting to get the fermionic representations for these examples.

# Appendix 1 Formulae involving $r(D)$

Let $r(n) = e^{T_{n-1}-T_n}$, $h(n) = e^{T_n}$. Then the operators $r(D), h(D)$, where $D = z\frac{d}{dz}$, have the simple properties

$$F_r(M, zz^*) := \left(1 + \sum_{n=1}^{\infty} e^{T_{M-n}-T_M} \frac{(zz^*)^{-n}}{n!}\right)(zz^*)^M = \tag{A1.1}$$

$$= \left(1 - \frac{1}{zz^*}\frac{r(D)}{D+a}\right)^{-a} \cdot (zz^*)^M = \left(1 - \frac{1}{zz^*}\frac{r(D)}{D+b}\right)^{-b} \cdot (zz^*)^M, \quad a, b \neq -M, \tag{A1.2}$$

$$\xi_r^{(\infty)}(\beta, z, D) := \sum_{n=1}^{\infty} \beta_n \left(\frac{1}{z} r(D)\right)^n, \quad \xi_r^{(0)}(\mathbf{t}, z, D) := \sum_{n=1}^{\infty} t_n \left(r(D) z\right)^n. \tag{A1.3}$$

Let us consider

$$f_r(M, \beta, z) := (zz^*)^M \left(1 + \sum_{n=1}^{\infty} e^{T_{M-n}-T_M} z^{-n} p_n(\beta)\right) = e^{\xi_r^{(\infty)}(\beta, z, D)} \cdot (zz^*)^M. \tag{A1.4}$$

Let $h(n) = e^{T_n}$. Then

$$\frac{h(D)}{h(0)} \cdot f_r(M, \beta, z) = (zz^*)^M e^{\xi(\beta, z^{-1})}, \quad \xi(\beta, z^{-1}) := \sum_{m=1}^{\infty} \beta_m z^{-m}. \tag{A1.5}$$

**Remark**. Let us put $m\beta_m = a(z^*)^{-m}, m = 1, 2, ...$ with some $a \neq 0$, then $f_r(M, \beta, z) = F_{r'}(M, zz^*)$, where $r'(D) = (D + a) r(D)$, since the elementary Schur polynomials (2.2.12) $p_n(\beta) = \frac{(a)_n}{n!}(z^*)^{-n}$. Eq.(3.12.7) generalizes this property for all partitions.

# Appendix 2 Orthogonal polynomials and matrix integrals

It is known that the hypergeometric functions (1.3.2) appear in the group representation theory and are connected with the so-called matrix integrals [14]. On the other hand the set of

examples [31, 30] reveals a connection between the matrix integrals [58] and the soliton theory. To establish this connection in our case, it is useful to consider the related systems of the orthogonal polynomials. Let us briefly describe how to write down these polynomials.

Let $M_+$ be the largest integer zero of $r$. Then the function

$$f_r^+(zz^*) = \sum_{n=0}^{+\infty} (zz^*)^{n+M_+} e^{T_{n+M_+}-T_{M_+}} \tag{A2.1}$$

is the eigenfunction of the operator $\frac{1}{z}r(D)$ with the eigenvalue $z^*$. Since operator $r(D)$ is invertible on the functions $\{z^M, M > M_+\}$ we write

$$f_r^+(zz^*) = \left(1 - \frac{1}{r(D)}zz^*\right)^{-1} \cdot (zz^*)^{M_+}. \tag{A2.2}$$

For example if we take $r(n) = n$ we obtain

$$f_r^+(zz^*) = e^{zz^*}. \tag{A2.3}$$

We use this function as weight function for a system of orthogonal polynomials $\{\pi_n^\pm, n = 0, 1, 2, \ldots\}$, related to the hypergeometric solution of KP:

$$\int_\gamma \pi_n^-(\mathbf{t}, \beta, z) e^{\xi(\mathbf{t},z)} f_r^+(zz^*) e^{\xi(\beta,z^*)} \pi_m^+(\mathbf{t}, \beta, z^*) dz dz^* = e^{-\phi_{M_++n}(\mathbf{t},\beta)} \delta_{n,m}. \tag{A2.4}$$

The corresponding two matrix integral [58] is the following one

$$\tau(M, \mathbf{t}, \beta) = \int e^{Tr\xi(\mathbf{t},Z)} f_r^+ \left(Tr(ZZ^*)\right) e^{Tr\xi(\beta,Z^*)} dZ dZ^*. \tag{A2.5}$$

Here $Z, Z^*$ are Hermitian $M \times M$ matrices.

# Appendix 3 The vertex operator action

Now we present relations between hypergeometric functions which follow from the soliton theory, for instance see [23],[22]. Let us introduce the operators which act on functions of $\mathbf{t}$ variables:

$$\Omega_r^{(\infty)}(\mathbf{t}^*) := -\frac{1}{2\pi i} \lim_{\epsilon \to 0} \oint V_\infty^*(z+\epsilon) \xi_r^{(\infty)}(\mathbf{t}^*, z, D) V_\infty(z) dz, \tag{A3.1}$$

$$\Omega_r^{(0)}(\mathbf{t}) := \frac{1}{2\pi i} \lim_{\epsilon \to 0} \oint V_0(z+\epsilon) \xi_r^{(0)}(\mathbf{t}, z, D) V_0^*(z) z^{-2} dz, \tag{A3.2}$$

where $V_\infty(z), V_\infty^*(z), V_0(z), V_0^*(z)$ are defined by (2.3.1). For instance

$$\Omega_r^{(\infty)}(\mathbf{t}^*) = \Omega_r^{(0)}(\mathbf{t}) = \sum_{n>0} n t_n t_n^*, \quad r = 1. \tag{A3.3}$$

The bosonization formulae [10],[15] give the relations which connect hypergeometric functions (3.1.7) and hypergeometric functions (4.1.10):

$$e^{-\Omega_{\tilde{r}1}^{(\infty)}(\tilde{\gamma}_1)} \cdots e^{-\Omega_{\tilde{r}k}^{(\infty)}(\tilde{\gamma}_k)} \cdots e^{\Omega_{rl}^{(\infty)}(\gamma_l)} \cdots e^{\Omega_{rl}^{(\infty)}(\gamma_l)} \tau_r(M, \mathbf{t}, \mathbf{t}^*) = \tau(M, \mathbf{t}, \mathbf{t}^*; \tilde{\gamma}, \gamma), \tag{A3.4}$$

$$e^{\Omega_{rl}^{(0)}(\gamma_l)} \cdots e^{\Omega_{rl}^{(0)}(\gamma_l)} \cdots e^{-\Omega_{\tilde{r}k}^{(0)}(\tilde{\gamma}_k)} \cdots e^{-\Omega_{\tilde{r}1}^{(0)}(\tilde{\gamma}_1)} \tau_r(M, \mathbf{t}, \mathbf{t}^*) = \tau(M, \mathbf{t}, \mathbf{t}^*; \tilde{\gamma}, \gamma). \tag{A3.5}$$

In particular we have shift argument formulae for the tau function (3.1.7):

$$e^{\Omega_r^{(\infty)}(\gamma)}\tau_r(M,\mathbf{t},\mathbf{t}^*)=\tau_r(M,\mathbf{t},\mathbf{t}^*+\gamma),\quad e^{\Omega_r^{(0)}(\gamma)}\tau_r(M,\mathbf{t},\mathbf{t}^*)=\tau_r(M,\mathbf{t}+\gamma,\mathbf{t}^*).\tag{A3.6}$$

Also we have

$$e^{\sum_{-\infty}^{\infty}\gamma_n Z_{nn}}\tau(M,\mathbf{t},\mathbf{T},\mathbf{t}^*)=e^{\sum_{-\infty}^{\infty}\gamma_n Z_{nn}^*}\tau(M,\mathbf{t},\mathbf{T},\mathbf{t}^*)=\tau(M,\mathbf{t},\mathbf{T}+\gamma,\mathbf{t}^*),\tag{A3.7}$$

where

$$Z_{nn}=-\frac{1}{4\pi^2}\oint\frac{z^n}{z^{*n}}V_\infty^*(z^*)V_\infty(z)dzdz^*,\quad Z_{nn}^*=-\frac{1}{4\pi^2}\oint\frac{z^n}{z^{*n}}V_0^*(z^*)V_0(z)dzdz^*.\tag{A3.8}$$

For instance

$$e^{\sum_{-\infty}^{\infty}T_n Z_{nn}}\exp\left(\sum_{n=1}^{\infty}nt_nt_n^*\right)=e^{\sum_{-\infty}^{\infty}T_n Z_{nn}^*}\exp\left(\sum_{n=1}^{\infty}nt_nt_n^*\right)=\tau(M,\mathbf{t},\mathbf{T},\mathbf{t}^*).\tag{A3.9}$$

# Appendix 4 Gauss factorization problem, additional symmetries, string equations and $\Psi DO$ on the circle

Let us describe relevant string equations following Takasaki and Takebe [19],[21]. We shall also consider this topic in a more detailed paper.

Let us introduce infinite matrices to describe KP and TL flows and symmetries, see [15]. Zakharov-Shabat dressing matrices are $K$ and $\bar{K}$. $K$ is a lower triangular matrix with unit main diagonal: $(K)_{ii}=1$. $\bar{K}$ is an upper triangular matrix. The matrices $K,\bar{K}$ depend on parameters $M,\mathbf{t},\mathbf{T},\mathbf{t}^*$. The matrices $(\Lambda)_{ik}=\delta_{i,k-1}$, $(\bar{\Lambda})_{ik}=\delta_{i,k+1}$. For each value of $\mathbf{t},\mathbf{T},\mathbf{t}^*$ and $M\in Z$ they solve Gauss (Riemann-Hilbert) factorization problem for infinite matrices:

$$\bar{K}=KG(M,\mathbf{t},\mathbf{T},\mathbf{t}^*),\quad G(M,\mathbf{t},\mathbf{T},\mathbf{t}^*)=\exp\left(\xi(\mathbf{t},\Lambda)\right)\Lambda^M G(\mathbf{0},\mathbf{T},\mathbf{0})\bar{\Lambda}^M\exp\left(\xi(\mathbf{t}^*,\bar{\Lambda})\right).\tag{A4.1}$$

We put $\log(\bar{K}_{ii})=\phi_{i+M}$, and a set of fields $\phi_i(\mathbf{t},\mathbf{t}^*),(-\infty<i<+\infty)$ solves the hierarchy of higher two-dimensional TL equations.

Take $L=K\Lambda K^{-1}$, $\bar{L}=\bar{K}\bar{\Lambda}\bar{K}^{-1}$, and $(\Delta)_{ik}=i\delta_{i,k}$, $\widehat{M}=K\Delta K^{-1}+M+\sum nt_nL^n$, $\widehat{\bar{M}}=\bar{K}\Delta\bar{K}^{-1}+M+\sum nt_n^*\bar{L}^n$. Then the KP additional symmetries [19],[21],[22],[23], [24] and higher TL flows [15] are written as

$$\partial_{\beta_n}K=-\left(\left(r(\widehat{M})L^{-1}\right)^n\right)_-K,\quad \partial_{\beta_n}\bar{K}=\left(\left(r(\widehat{M})L^{-1}\right)^n\right)_+\bar{K},\tag{A4.2}$$

$$\partial_{t_n^*}K=-\left(\bar{L}^n\right)_-K,\quad \partial_{t_n^*}\bar{K}=\left(\bar{L}^n\right)_+\bar{K}.\tag{A4.3}$$

Then the string equations are

$$\bar{L}L=r(\widehat{M}),\tag{A4.4}$$

$$\widehat{\bar{M}}=\widehat{M}.\tag{A4.5}$$

The first equation is a manifestation of the fact that the group time $\beta_1$ of the additional symmetry of KP can be identified with the Toda lattice time $t_1^*$. In terms of tau-function we have the equation (A3.6) in terms of vertex operator action [22],[23], or the equations (3.7.1) in case the tau-function is written in Miwa variables.

The second string equation (A4.5) is related to the symmetry of our tau-functions with respect to $\mathbf{t} \leftrightarrow \beta$.

When

$$r(M) = M + a, \tag{A4.6}$$

the equations (A4.4),(A4.5) describe $c = 1$ string , see [20],[21]. In this case we easily get the relation

$$[\bar{L}, L] = 1. \tag{A4.7}$$

The string equations in the form of Takasaki allows us to notice the similarity to the different problem. The dispersionless limit of (A4.7) (and also of (A4.5), (A4.4), where $r(M) = M^n$, and of (A4.6)) will be written as

$$\bar{\lambda}\lambda = \mu^n, \quad n \in Z, \tag{A4.8}$$

$$\bar{\mu} = \mu. \tag{A4.9}$$

The case when $\bar{\lambda}$ and $\bar{\mu}$ are complex conjugate of $\lambda$ and of $\mu$ respectively, is of interest. These string equations (mainly the case $n = 1$) were recently investigated to solve the so-called Laplacian growth problem, see [41]. We are grateful to A.Zabrodin for the discussion on this problem. For the dispersionless limit of the KP and TL hierarchies see [19].

In case the function $r(n)$ has zeroes (described by divisor $\mathbf{m} = (M_1, ...)$: $r(M_k) = 0$), one needs to produce the replacement:

$$\Lambda \to \Lambda(\mathbf{m}), \quad \bar{\Lambda} \to \bar{\Lambda}(\mathbf{m}), \tag{A4.10}$$

where new matrices $\Lambda(\mathbf{m})$,$\bar{\Lambda}(\mathbf{m})$ are defined as

$$(\Lambda(\mathbf{m}))_{i,j} = \delta_{i,j-1}, j \neq M_k, \quad (\Lambda(\mathbf{m}))_{i,j} = 0, j = M_k, \tag{A4.11}$$

$$(\bar{\Lambda}(\mathbf{m}))_{i,j} = \delta_{i,j+1}, i \neq M_k, \quad (\bar{\Lambda}(\mathbf{m}))_{i,j} = 0, i = M_k. \tag{A4.12}$$

This modification describes the open TL equation (3.4.9):

$$\partial_{t_1}\partial_{\beta_1}\varphi_n = \delta(n)e^{\varphi_{n-1}-\varphi_n} - \delta(n+1)e^{\varphi_n-\varphi_{n+1}}. \tag{A4.13}$$

The set of fields $\phi_\infty, \dots, \phi_{M_1}, \phi_{M_1+1} \dots$ consists of the following parts due to the conditions

$$\sum_{n=-\infty}^{M_s} \phi_n = 0, \qquad \sum_{n=M_k+1}^{M_{k+1}} \phi_n = 0, \qquad \sum_{n=\infty}^{M_1+1} \phi_n = 0, \tag{A4.14}$$

which result from $\tau(M_k, \mathbf{t}, \mathbf{t}^*) = 1$.

Each internal part (3.5.3) of this Toda chain is a $M_{k+1} - M_k$ sites chain. There are two semiinfinite parts which correspond to (3.5.2) and (3.5.4).

**Remark 5** *The matrix $r(\widehat{M})$ contains $(\widehat{M} - b_i)$ in the denominator. The matrix $(\widehat{M} - b_i)^{-1}$ is $K(\Delta - b_i)^{-1}K^{-1}(1 + O(\mathbf{t}))$ (compare the consideration of the inverse operators with [24]).*

The KP tau-function (3.1.7) can be obtained as follows.

$$G(M, \mathbf{t}, \mathbf{T}, \mathbf{t}^*) = G(\mathbf{0}, \mathbf{T}, \mathbf{0})U(M, \mathbf{t}, \beta), \quad U(M, \mathbf{t}, \beta) = U^+(\mathbf{t})U^-(M, \beta). \tag{A4.15}$$

$$U^+(\mathbf{t}) = \exp\left(\xi(\mathbf{t}, \Lambda)\right), \quad U^-(M, \beta) = \exp\left(\xi(\beta, \Lambda^{-1}r\left(\Delta + M\right))\right), \tag{A4.16}$$

The matrix $G(\mathbf{0}, \mathbf{T}, \mathbf{0})$ is related to the transformation of the eq.(3.3.11) to the eq.(3.3.8). By taking the projection [15] $U \mapsto U_{--}$ for nonpositive values of matrix indices we obtain a determinant representation of the tau-function (3.1.7):

$$\tau_r(M, \mathbf{t}, \beta) = \frac{\det U_{--}(M, \mathbf{t}, \beta)}{\det \left(U^+_{--}(\mathbf{t})\right) \det \left(U^-_{--}(M, \beta)\right)} = \det U_{--}(M, \mathbf{t}, \beta), \tag{A4.17}$$

since both determinants in the denominator are equal to one. Formula (A4.17) is also a Segal-Wilson formula for $GL(\infty)$ 2-cocycle [55] $C_M\left(U^+(-\mathbf{t}), U^-(-\beta)\right)$. Choosing the function $r$ as in *Section 3.2* we obtain hypergeometric functions listed in the *Introduction*.

**Remark 6** *Therefore the hypergeometric functions which were considered above have the meaning of $GL(\infty)$ two-cocycle on the two multiparametrical group elements $U^+(\mathbf{t})$ and $U^-(M, \beta)$. Both elements $U^+(\mathbf{t})$ and $U^-(M, \beta)$ can be considered as elements of group of pseudodifferential operators on the circle. The corresponding Lie algebras consist of the multiplication operators $\{z^n; n \in N_0\}$ and of the pseudodifferential operators $\{\left(\frac{1}{z} r(z\frac{d}{dz} + M)\right)^n; n \in N_0\}$. Two sets of group times $\mathbf{t}$ and $\beta$ play the role of indeterminates of the hypergeometric functions (3.10.5). Formulas (3.1.7) and (3.1.12) mean the expansion of $GL(\infty)$ group 2-cocycle in terms of corresponding Lie algebra 2-cocycle*

$$c_M(z, \frac{1}{z} r(D)) = r(M), \quad c_M(\tilde{r}(D)z, \frac{1}{z} r(D + M)) = \tilde{r}(M) r(M). \tag{A4.18}$$

*Japanese cocycle is cohomological to Khesin-Kravchenko cocycle [56] for the $\Psi DO$ on the circle:*

$$c_M \sim c_0 \sim \omega_M, \tag{A4.19}$$

*which is*

$$\omega_M(A, B) = \oint res_\partial A[\log(D + M), B] dz, \quad A, B \in \Psi DO. \tag{A4.20}$$

*For the group cocycle we have*

$$C_M\left(e^{-\sum z^n t_n}, e^{-\sum (z^{-1} r(D))^n \beta_n}\right) = \tau_r(M, \mathbf{t}, \beta), \tag{A4.21}$$

*where we imply that the order of $\Psi DO$ $r(D)$ is 1 or less. About properties of $e^{-\sum (z^{-1} r(D))^n \beta_n}$ see (A1.3),(A1.4).*

**Remark 7** *It is interesting to note that in case of hypergeometric functions ${}_pF_s$ (1.2.6) the order of $r$ is $p - s$ (see* Example 3*), and the condition $p - s \le 1$ is the condition of the convergence of this hypergeometric series, see [13]. Namely the radius of convergence is finite in case $p - s = 1$, it is infinite when $p - s < 1$ and it is zero for $p - s > 1$ (this is true for the case when no one of $a_k$ in (1.2.6) is nonnegative integer).*

**Remark 8** *The set of functions $\{w(n, z), n = M, M + 1, M + 2, \ldots\}$, where*

$$w(n, z) = \exp\left(-\sum_{m=0}^{\infty} t^*_m \left(\frac{1}{z} r(D)\right)^m\right) \cdot z^n, \tag{A4.22}$$

*may be identified with Sato Grassmannian (3.11.5) related to the cocycle (A4.21). The dual Grassmannian (3.11.6) is the set of one forms $\{w^*(n, z), n = M, M + 1, M + 2, \ldots\}$,*

$$w^*(n, z) = \exp\left(\sum_{m=0}^{\infty} t^*_m \left(\frac{1}{z} r(-D)\right)^m\right) \cdot z^{-n} dz. \tag{A4.23}$$

# Appendix 5 Equations with respect to T variables

Let us show that variables $\mathbf{T}$ play the role of time variables. We shall write down some equation involving the differentiation with respect to $\mathbf{T}$. The relevant Liouville equations is constructed in terms of

$$e^{v_k} = \frac{\langle M+1|e^{H(\mathbf{t})}\psi_k e^{H_0(\mathbf{T})}e^{H^*(\mathbf{t}^*)}|M\rangle\langle M-1|e^{H(\mathbf{t})}\psi_k^* e^{H_0(\mathbf{T})}e^{H^*(\mathbf{t}^*)}|M\rangle}{\langle M|e^{H(\mathbf{t})}e^{H_0(\mathbf{T})}e^{H^*(\mathbf{t}^*)}|M\rangle^2}. \tag{A5.1}$$

The equations are

$$-\frac{\partial^2 v_k}{\partial t_1 \partial T_k} = e^{v_k}, \quad k \in Z. \tag{A5.2}$$

To get three-wave equations take

$$\beta_{nm} = \frac{\langle M|e^{H(\mathbf{t})}\psi_m\psi_n^* e^{H_0(\mathbf{T})}e^{H^*(\mathbf{t}^*)}|M\rangle}{\langle M|e^{H(\mathbf{t})}e^{H_0(\mathbf{T})}e^{H^*(\mathbf{t}^*)}|M\rangle}, \tag{A5.3}$$

$$\beta_{1'n} = \frac{\langle M+1|e^{H(\mathbf{t})}\psi_n e^{H_0(\mathbf{T})}e^{H^*(\mathbf{t}^*)}|M\rangle}{\langle M|e^{H(\mathbf{t})}e^{H_0(\mathbf{T})}e^{H^*(\mathbf{t}^*)}|M\rangle}, \quad \beta_{n1'} = \frac{\langle M-1|e^{H(\mathbf{t})}\psi_n^* e^{H_0(\mathbf{T})}e^{H^*(\mathbf{t}^*)}|M\rangle}{\langle M|e^{H(\mathbf{t})}e^{H_0(\mathbf{T})}e^{H^*(\mathbf{t}^*)}|M\rangle}. \tag{A5.4}$$

Then we obtain

$$-\partial_{T_n}\beta_{1'm} = \beta_{1'n}\beta_{nm}, \quad -\partial_{T_n}\beta_{m1'} = \beta_{mn}\beta_{n1'}, \quad \partial_{t_1}\beta_{mn} = \beta_{m1'}\beta_{1'n}, \quad m \neq n, \quad n,m \in Z. \tag{A5.5}$$

One obtains these equations with the help of the Lax type representation [57]:

$$[\partial_{T_m} + \beta_{1'm}\partial_{t_1}^{-1}\beta_{m1'}, \partial_{T_n} + \beta_{1'n}\partial_{t_1}^{-1}\beta_{n1'}] = 0. \tag{A5.6}$$

It is possible to write down the discrete versions of these equations, and (the discrete and the continues) equations involving only $\mathbf{T}$ variables.

# Appendix 6 Orthogonal $q$-polynomials

Now we present the fermionic representation of polynomials listed in the *Introduction*. These polynomials are obtained by the specification of (3.10.25).
**$q$-Askey-Wilson polynomials** are defined as

$$p_n(x;a,b,c,d|q) = a^{-n}(ab;q)_n(ac;q)_n(ad;q)_n$$
$$\times {}_4\varphi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, ae^{i\eta}, ae^{-i\eta} \\ ab, \quad ac, \quad ad \end{matrix}\middle| q, q\right), \qquad x = \cos\eta. \tag{A6.1}$$

Let operator ${}_4r_3^{(q)}(D)$ be

$${}_4r_3^{(q)}(D) = \frac{(1-q^{-n+D})(1-abcdq^{n-1+D})(1-ae^{i\eta}q^D)(1-ae^{-i\eta}q^D)}{(1-abq^D)(1-acq^D)(1-adq^D)}. \tag{A6.2}$$

For this operator we have:

$$\frac{{}^4\tau^3(M,\mathbf{t},\mathbf{T},\mathbf{t}^*)}{{}^4\tau^3(M,\mathbf{0},\mathbf{T},\mathbf{0})} = \langle M|e^{H(\mathbf{t})}e^{-A(\mathbf{t}^*)}|M\rangle, \tag{A6.3}$$

where

$$A_k = \frac{1}{2\pi i}\oint \psi^*(z)\left(\frac{1}{z}{}_4r_3^{(q)}(D)\right)^k \psi(z), \quad k = 1, 2, \ldots. \tag{A6.4}$$

If we take the variables $\mathbf{t}^*$ as in (3.10.17) and

$$\mathbf{t} = (q, \frac{q^2}{2}, \frac{q^3}{3}, \ldots), \tag{A6.5}$$

$$\frac{{}_4\tau_3^{(q)}(M, \mathbf{t}, \mathbf{T}, \mathbf{t}^*)}{{}_4\tau_3^{(q)}(M, \mathbf{0}, \mathbf{T}, \mathbf{0})} = {}_4\varphi_3\left(\begin{matrix} q^{M-n}, q^{M+n-1}abcd, aq^M e^{i\eta}, aq^M e^{-i\eta} \\ q^M ab, \quad q^M ac, \quad q^M ad \end{matrix}\middle| q, q\right) =$$
$$\sum_{m=0}^{+\infty} \frac{(q^{M-n};q)_m(q^{M+n-1}abcd;q)_m(aq^Me^{i\eta};q)_m(aqe^{-i\eta};q)_m}{(abq^M;q)_m(acq^M;q)_m(adq^M;q)_m}\frac{q^m}{(q;q)_m}. \tag{A6.6}$$

For $M = 0$ we obtain $q$-Askey-Wilson polynomials:

$$p_n(x; a, b, c, d|q) = aq^{-n}(ab;q)_n(ac;q)_n(ad;q)_n \frac{{}^4\tau^3(0, \mathbf{t}, \mathbf{T}, \mathbf{t}^*)}{{}^4\tau^3(0, \mathbf{0}, \mathbf{T}, \mathbf{0})}. \tag{A6.7}$$

If parameters in (A6.2) are

$$a = q^{\frac{2\alpha+1}{4}}, \quad b = -q^{\frac{2\nu+1}{4}}, \quad c = q^{\frac{2\alpha+3}{4}}, \quad d = -q^{\frac{2\nu+3}{4}}, \tag{A6.8}$$

we get a fermionic representation for continuous $q$-**Jacobi polynomials**

$$P_n^{(\alpha,\nu)}(x|q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n)}\frac{{}^4\tau^3(0, \mathbf{t}, \mathbf{T}, \mathbf{t}^*)}{{}^4\tau^3(0, \mathbf{0}, \mathbf{T}, \mathbf{0})}. \tag{A6.9}$$

For $c = -a, b = -d = q^{\frac{1}{2}}a$ we have $q$-**Gegenbauer polynomials**

$$C_n(\cos\eta; \mu|q) = \frac{(\mu^2;q)_n}{\mu^{\frac{n}{2}}(q;q)_n}\frac{{}^4\tau^3(0, \mathbf{t}, \mathbf{T}, \mathbf{t}^*)}{{}^4\tau^3(0, \mathbf{0}, \mathbf{T}, \mathbf{0})}, \quad \mu = a^2. \tag{A6.10}$$

**Clebsch-Gordan coefficients** $C_q(\mathbf{l}, \mathbf{j})$ see [14].
Let

$${}_3r_2^{(q)}(D) = \frac{(1-q^{j-l_1+D})(1-q^{l_1+j+1+D})(1-q^{-l+m+D})}{(1-q^{l_2-l+j+1+D})(1-q^{-l-l_2+j+D})}. \tag{A6.11}$$

For variables from (3.10.17) and (A6.5) we have

$$\frac{{}_3\tau_2^{(q)}(M, \mathbf{t}, \mathbf{T}, \mathbf{t}^*)}{{}_3\tau_2^{(q)}(M, \mathbf{0}, \mathbf{T}, \mathbf{0})} = {}_3\Phi_2\left(\begin{matrix} j-l_1+M, l_1+j+1+M, -l+m+M \\ l_2-l+1+M, -l-l_2+j+M \end{matrix}\middle| q, q\right). \tag{A6.12}$$

Thus we have the fermionic representation

$$C_q(\mathbf{l}, \mathbf{j}) = \frac{(-1)^{l_1-j}q^B\Delta(\mathbf{l})[l+l_2-j]!([\mathbf{l}, \mathbf{j}][2l+1])^{\frac{1}{2}}}{[l_1-l_2+l]![l+l_2-l_1]![l_2-l+j]![l_1-j]![l_2+k]![l-m]!}\frac{{}_3\tau_2^{(q)}(0, \mathbf{t}, \mathbf{T}, \mathbf{t}^*)}{{}_3\tau_2^{(q)}(0, \mathbf{0}, \mathbf{T}, \mathbf{0})}. \tag{A6.13}$$

$q$-**Hahn polynomials**
Let us take the operator:

$${}_3r_2^{(q)}(D) = \frac{(1-q^{-n+D})(1-abq^{n+1+D})(1-q^{-x+D})}{(1-aq^{D+1})(1-q^{D-N})}, \quad n \leq N. \tag{A6.14}$$

The corresponding tau function whose variables are defined in (3.10.17) and (A6.5):

$$\frac{{}_3\tau_2^{(q)}(M,\mathbf{t},\mathbf{T},\mathbf{t}^*)}{{}_3\tau_2^{(q)}(M,\mathbf{0},\mathbf{T},\mathbf{0})} = {}_3\varphi_2\left(\begin{matrix} q^{M-n}, abq^{M+n+1}, q^{M-x} \\ aq^{M+1}, \quad q^{M-N} \end{matrix} \,\middle|\, q,q\right). \tag{A6.15}$$

Therefore the fermionic representation of $q$-Hahn polynomials is

$$Q_n(q^{-x};a,b;-N|q) = \frac{{}_3\tau_2^{(q)}(0,\mathbf{t},\mathbf{T},\mathbf{t}^*)}{{}_3\tau_2^{(q)}(0,\mathbf{0},\mathbf{T},\mathbf{0})}. \tag{A6.16}$$

**$q$-Racah polynomials** Now we take as $r(D)$

$${}_4r_3^{(q)}(D) = \frac{(1-q^{-n+D})(1-abq^{n+1+D})(1-q^{-x+D})(1-cdq^{x+1+D})}{(1-aq^{D+1})(1-bdq^{D+1})(1-cq^{D+1})}. \tag{A6.17}$$

The tau function is:

$$\frac{{}_4\tau_3^{(q)}(M,\mathbf{t},\mathbf{T},\mathbf{t}^*)}{{}_4\tau_3^{(q)}(M,\mathbf{0},\mathbf{T},\mathbf{0})} = {}_4\varphi_3\left(\begin{matrix} q^{M-n}, abq^{M+n+1}, q^{M-x}, cdq^{M+x+1} \\ aq^{M+1}, \quad bdq^{M+1}, \quad cq^{M+1} \end{matrix} \,\middle|\, q,q\right), \tag{A6.18}$$

where $\mathbf{t}^*$ and $\mathbf{t}$ as in (3.10.17) and (A6.5). Thus we have an expression

$$R_n(\mu(x);a,b,c,d|q) = \frac{{}_4\tau_3^{(q)}(0,\mathbf{t},\mathbf{T},\mathbf{t}^*)}{{}_4\tau_3^{(q)}(0,\mathbf{0},\mathbf{T},\mathbf{0})}, \quad \mu(x) = q^{-x} + cdq^{x+1}. \tag{A6.19}$$

**Little $q$-Jacobi polynomials**
Setting operator $r(D)$:

$${}_2r_1^{(q)}(D) = \frac{(1-q^{-n+D})(1-abq^{n+1+D})}{(1-aq^{D+1})}, \tag{A6.20}$$

we get tau function:

$$\frac{{}_2\tau_1^{(q)}(M,\mathbf{t},\mathbf{T},\mathbf{t}^*)}{{}_2\tau_1^{(q)}(M,\mathbf{0},\mathbf{T},\mathbf{0})} = {}_2\varphi_1\left(\begin{matrix} q^{M-n}, abq^{M+n+1} \\ aq^{M+1} \end{matrix} \,\middle|\, q,qx\right), \tag{A6.21}$$

where $t_m = \frac{(qx)^m}{m}$ and $\mathbf{t}^*$ as in (3.10.17).

$$p_n(x;a,b|q) = \frac{{}_2\tau_1^{(q)}(0,\mathbf{t},\mathbf{T},\mathbf{t}^*)}{{}_2\tau_1^{(q)}(0,\mathbf{0},\mathbf{T},\mathbf{0})}. \tag{A6.22}$$

# Appendix 7

Since we have $\exp\left(T_n : \psi_n^*\psi_n :\right) = 1 + \left(e^{T_n}-1\right) : \psi_n^*\psi_n :$ the tau-function (3.3.1) is a linear function of each $e^{T_m}-1$. Take in the series (3.3.1) a coefficient before the monomial $\prod_i(e^{T_{m_i}}-1)$, where $\{m_i\}$ form a set of indices $X$. Denote this coefficient as $\rho(X,\mathbf{t},\mathbf{t}^*)\exp\left(-\sum_{k=1}^\infty kt_kt_k^*\right)$. The function $\rho(X,\mathbf{t},\mathbf{t}^*)$ has the following meaning in the probability theory [60]. The Schur measure on partitions is defined via the formula:

$$\mu(\mathbf{n},\mathbf{t},\mathbf{t}^*) = s_{\mathbf{n}}(\mathbf{t})s_{\mathbf{n}}(\mathbf{t}^*)\exp\left(-\sum_{k=1}^\infty kt_kt_k^*\right), \tag{A7.1}$$

where $\mathbf{n}$ is a partition, and $\mathbf{t}, \mathbf{t}^*$ are parameters. Then the function $\rho(X, \mathbf{t}, \mathbf{t}^*)$ describes the probability that the set $\{n_i - i\}$ contains a given set $X \subset Z$. The Schur measure has a natural group-theoretical interpretation, see [60]. In this context it may be of interest to consider the following measure on partitions, which generalizes (A7.1):

$$\mu_{\tilde{r}r}(\mathbf{n}, \mathbf{t}, \mathbf{t}^*) = \frac{\tilde{r}_{\mathbf{n}}(M) r_{\mathbf{n}}(M) s_{\mathbf{n}}(\mathbf{t}) s_{\mathbf{n}}(\mathbf{t}^*)}{\langle M | e^{\tilde{A}(\mathbf{t})} e^{A(\mathbf{t}^*)} | M \rangle}, \tag{A7.2}$$

see (3.1.12). Let us note that in the case of hypergeometric tau-functions, function $r_{\mathbf{n}}$ is a rational function of the Schur polynomials, see (3.12.5),(3.12.6) below.

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